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Quantum tachyons

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Abstract. The interaction of superluminal radiation with matter in atomic bound-bound and bound-free transitions is investigated. We study transitions in the relativistic hydrogen atom effected by superluminal quanta. The superluminal radiation field is coupled by minimal substitution to the Dirac equation in a Coulomb potential. We quantize the interaction to obtain the transition matrix for induced and spontaneous superluminal radiation in hydrogen-like ions. The tachyonic photoelectric effect is scrutinized, the cross-sections for ground state ionization by transversal and longitudinal tachyons are derived. We examine the relativistic regime, high electronic ejection energies, as well as the first order correction to the non-relativistic cross-sections. In the ultra-relativistic limit, both the longitudinal and transversal cross-sections are peaked at small but noticeably different scattering angles. In the non-relativistic limit, the longitudinal cross-section has two maxima, and its minimum is located at the transversal maximum. Ionization cross-sections can thus be used to discriminate longitudinal radiation from transversal tachyons and photons.

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1 Introduction

When considering superluminal quanta, we may try a wave theory or a particle picture as starting point. The latter has been studied for quite some time, but did not result in viable interactions with matter [1–16]. Thus we suggest to model tachyons as quantized wave fields with negative mass-square, coupled by minimal substitution to subluminal particles. Interaction with matter is indeed the crucial point, after all, what else can one expect from a theory of tachyons other than suggestions as to where to search for them? We will maintain the best established interaction mechanism, minimal substitution, by treating tachyons like photons with negative mass-square, a real Proca field minimally coupled to matter [17–20].

The superluminal energy flux can be split into a transversal and a longitudinal component, and the different polarizations are quantized in different statistics to obtain a positive definite energy operator; transversal quanta are bosonic, longitudinal ones are fermionic. The spin-statistics theorem and most other quantum field theoretic no-go theorems are not applicable outside the lightcone, as they assume microcausality, which means, in a relativistic context, the non-existence of superluminal signal transfer [21,22]. This theory of superluminal radiation is non-relativistic, invoking the absolute cosmic spacetime, cf. [23,24] and Section 4 for more discussion on the underlying spacetime view.

We will consider hydrogenic systems and derive the T-matrix and Einstein coefficients for tachyonic bound-bound transitions. The superluminal radiation modes, minimally coupled to the Dirac field in a Coulomb potential, can be treated perturbatively in linear order, due to the very small tachyonic interaction constant, the ratio of tachyonic and electric fine structure constants being $\alpha_q/\alpha_e \approx 1.4 \times 10^{-11}$, cf. [20]. The electromagnetic second order contribution overpowers the tachyonic counterpart by some twenty-two orders, so that elementary statistical procedures such as detailed balancing are sufficient for the second quantization of the interaction. Linearization on account of the tiny interaction will be used throughout, there is no need to develop a perturbation theory beyond the linear order. One can also reckon that the Lagrangian of the Proca field is itself just the linearization of a non-linear Born-Infeld type of Lagrangian, as this seems to be the most straightforward way to achieve a finite classical self-energy [25,26].

We will investigate superluminal bound-free transitions, in particular the cross-sections for tachyonic ionization of hydrogen-like ground states. We will study the relativistic regime, high electronic ejection energies, especially the ultra-relativistic limit, large electronic Lorentz factors. The ionizing superluminal quanta can be transversally or longitudinally polarized, and we will determine the angular extrema of the respective cross-sections. The scattering angles at which the extrema occur crucially depend on the polarization of the incident radiation, which can be used to disentangle longitudinal and transversal quanta.

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In Section 2, we will set up the formalism, discuss the Dirac equation coupled to the Proca field, the spectral decomposition of the Dirac Hamiltonian, and the current matrix. We will mainly work with 2-spinors, refraining from manifest covariance at an early stage. In conjunction is Appendix A, where we examine the non-relativistic limit, the Pauli equation coupled to the tachyon field, and calculate matrix elements of the helicity operator.

In Section 3, we quantize the free tachyon field and the interaction Hamiltonian. The transition matrix for tachyonic bound-bound transitions in the relativistic hydrogen atom is derived. We calculate the tachyonic ionization cross-sections of the ground state, taking into account electron spin and relativistic ejection energies. We study transversal and longitudinal ionization as well as the corresponding recombination cross-sections. The total cross-sections are compared with the electromagnetic counterpart, photoionization. We consider the ultra-relativistic limit and determine the angular maxima of the transversal and longitudinal cross-sections. We also derive the first order correction to the non-relativistic limit in Born approximation. In this limit, the differences in the peak structure of the transversal and longitudinal cross-sections are even more pronounced than in the ultra-relativistic regime, the transversal maximum coinciding with the longitudinal minimum. In Appendix B, we calculate the matrix elements needed for the cross-sections and perform the spin averaging. In Section 4, we present our conclusions.

2 Superluminal radiation fields coupled to spinor currents

We set up the Lagrange formalism to deal with superluminal radiation modes coupled by minimal substitution to spinor fields. We introduce the notation used in the actual calculations carried out in the next section, specifying sign conventions, units, and normalizations; derivations will mostly be skipped. The formalism is kept elementary and very explicit by employing 2-spinors and Pauli matrices at an early stage in the spectral analysis. Non-relativistic spinor currents coupled to the tachyon field are studied in Appendix A.

The superluminal radiation field satisfies the Proca equation, $F^{\mu\nu}{}_{,\nu} - m_t^2 A^\mu = c^{-1} j^\mu$, where $F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$, and j^α is the subluminal current specified below. The negative mass-square, $-m_t^2$, makes the wave propagation superluminal, and the tachyon mass m_t is a shortcut for $m_t c/\hbar$. This field equation can be derived from the Lagrangian $L_P + L_{int}$, where $L_P = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_t^2 A_\mu A^\mu$ and $L_{int} = c^{-1}A_\mu j^\mu$. Greek indices run from 0 to 3, Latin ones from 1 to 3, the sign convention for the metric is $\eta_{\mu\nu} = \mathrm{diag}(-c^2, 1, 1, 1)$.

The subluminal matter field satisfies the Dirac equation, $\gamma^\mu \nabla_\mu^A \psi + (mc/\hbar)\psi = 0$, the spinor being coupled by minimal substitution to the tachyonic vector potential,

$$L_\psi = -i\hbar c^2 \left(\frac{1}{2}\bar{\psi}\gamma^\mu \nabla_\mu^A \psi - \frac{1}{2}\left(\nabla_\mu^{A*}\bar{\psi}\right)\gamma^\mu \psi + \frac{mc}{\hbar}\psi\bar{\psi} \right),$$

$$\nabla_\mu^A = \partial_\mu - i\tilde{e}A_\mu^{em} - i\tilde{q}A_\mu, \quad \tilde{q} = q/(\hbar c), \quad \tilde{e} = e/(\hbar c). \tag{2.1}$$

The asterisk in ∇_μ^{A*} indicates complex conjugation, $\psi^\dagger$ means transposition and complex conjugation (also for 2-spinors and matrices), and $\bar{\psi} = \psi^\dagger \gamma^0$. We have also included an external electromagnetic potential A_μ^{em} in ∇_μ^A, which will be treated non-perturbatively, and we will occasionally use the shortcut $\nabla_\mu^{em} := \partial_\mu - i\tilde{e}A_\mu^{em}$. The electric and tachyonic charges carried by the spinor field are denoted by e and q, respectively. We use the Heaviside-Lorentz system, so that $\alpha_e = e^2/(4\pi\hbar c) \approx 1/137$ and $\alpha_q = q^2/(4\pi\hbar c) \approx 1.0 \times 10^{-13}$ are the electric and tachyonic fine structure constants. We note the ratio $\alpha_q/\alpha_e \approx 1.4 \times 10^{-11}$, the tachyon mass $m_t \approx m/238 \approx 2.15\ \mathrm{keV/c^2}$, and the inverse Compton wave length $m_t c/\hbar \approx 1.09 \times 10^8\ \mathrm{cm}^{-1}$. These estimates are obtained from hydrogenic Lamb shifts [20].

The coupling to an attractive Coulomb potential, $V = -Ze^2/(4\pi r)$, is effected by the derivatives $\nabla_0^A = \partial_0 + i\hbar^{-1}V - i\tilde{q}A_0$ and $\nabla_k^A = \partial_k - i\tilde{q}A_k$. Identifying the Dirac Lagrangian L_ψ with L_{int}, we find the current in the Proca equation, $j^\mu = -qc^2\bar{\psi}\gamma^\mu\psi$. We will use a set of Dirac matrices, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$, in standard representation,

$$\gamma^0 = \frac{1}{ic}\begin{pmatrix} \mathrm{id} & 0 \\ 0 & -\mathrm{id} \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & -i\sigma^k \\ i\sigma^k & 0 \end{pmatrix}, \tag{2.2}$$

so that the charge density reads $\rho = j^0 = q\psi^\dagger\psi$. The Pauli matrices σ^k are listed in (A.2), and this representation will be used throughout, without further mentioning. The Lagrangian L_ψ in (2.1) has the dimension of an energy density if $\psi \sim L^{-3/2}$, where L is the "box size". We split the Dirac equation into two coupled 2-spinor equations,

$$i\hbar\nabla_0^A \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = -i\hbar c\sigma^k \nabla_k^A \begin{pmatrix} \chi \\ \varphi \end{pmatrix} + mc^2 \begin{pmatrix} \varphi \\ -\chi \end{pmatrix}, \tag{2.3}$$

which is apparently effected by the substitution $\psi = (\varphi, \chi)^{\mathrm{t}}$ in the manifestly covariant version of the Dirac equation as stated above. The superscript "t" stands for transposition, here just for typographical convenience. This decomposition of the wave function into 2-spinors φ and χ will extensively be used in the following, always in conjunction with the standard representation (2.2). For instance, the spinor current reads,

$$j^0 = q(\varphi^\dagger\varphi + \chi^\dagger\chi), \quad j^k = qc(\chi^\dagger\sigma^k\varphi + \varphi^\dagger\sigma^k\chi). \tag{2.4}$$

We will need some more matrices of the Dirac algebra,

$$\gamma^5 = ic\gamma^0\gamma^1\gamma^2\gamma^3, \quad \beta = ic\gamma^0, \quad \alpha^k = -c\gamma^0\gamma^k. \tag{2.5}$$

We may write $\gamma^5 = \sigma_1$ and $c\gamma^0\gamma^5 = \sigma_2$, as well as $\beta = \sigma_3$. Here, exceptionally, the Pauli matrices σ_k (as defined in (A.2)) are to be understood as four-by-four matrices, that is, their coefficients have to be multiplied by the two-by-two unit matrix "id", cf. (2.2). Moreover,

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad i\gamma^k\gamma^5 = \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix},$$

$$\alpha^k\gamma^5 = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \tag{2.6}$$

and γ^5 anticommutes with γ^μ. The 3-vectors α^k and σ^k will occasionally be denoted by $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$. All matrices defined in (2.2), (2.5) and (2.6) are hermitian, except for the anti-hermitian γ^0. The operator $\boldsymbol{\Sigma} := i\gamma^k\gamma^5$ will be used to define spin and helicity.

We multiply the Dirac equation with γ^0, and identify the Hamiltonian by means of $i\hbar\psi_{,t} = (H^{em} + H^{int})\psi$, where

$$H^{em} = -i\hbar c\alpha^k\nabla_k^{em} + mc^2\beta - ec^{-1}A_0^{em},$$
$$H^{int} = -q(\alpha^k A_k + c^{-1}A_0). \tag{2.7}$$

The matrices α^k and β are defined in (2.6) and (2.5). We put $\hbar = c = 1$, and consider the free Dirac Hamiltonian, $H_0 = -i\alpha^i\partial_i + m\beta$. In the free Dirac equation, $i\hbar\psi_{,t} = H_0\psi$, we use the separation ansatz $\psi = ue^{i\varepsilon(\mathbf{kx}-\omega_k t)}$, where $\varepsilon = \pm 1$ stands for positive/negative frequency solutions. The frequencies ω_k are defined positive throughout, for particles and antiparticles alike. In momentum space, we thus arrive at the eigenvalue equation $\hat{H}_0 u = \varepsilon\omega_k u$, where $\hat{H}_0 = (\varepsilon\boldsymbol{\alpha}\mathbf{k} + m\beta)$. The dispersion relation, $\omega_k = \sqrt{k^2+m^2}$, emerges as determinant condition. In the rest frame, $\mathbf{k} = 0$, $\omega_k = m$, we find the solutions $u' = \sqrt{m/\omega_k}(\varphi', 0)^{\mathrm{t}}$ if $\varepsilon = 1$, and $u' = \sqrt{m/\omega_k}(0, \chi')^{\mathrm{t}}$ for $\varepsilon = -1$, where the 2-spinors $\varphi'$and χ' can be chosen arbitrarily. We will adopt the arbitrary rest frame normalization $u'^{\dagger}u' = m/\omega_k$. Primed spinors and operators always refer to the rest frame. The helicity operator, $\mathbf{k}\boldsymbol{\Sigma} := -i\gamma^5\gamma^m k_m$, cf. (2.6), commutes with $\hat{H}_0' = m\beta$, so that we can simultaneously diagonalize $\hat{H}_0' u' = \varepsilon m u'$ and $\mathbf{k}\boldsymbol{\Sigma}u' = ksu'$ in the rest frame. The latter equation reduces to $(\boldsymbol{\sigma}\mathbf{k} - ks)\varphi'_{k,s} = 0$ for $\varepsilon = 1$ and to $(\boldsymbol{\sigma}\mathbf{k}+ks)\chi'_{k,s} = 0$ for $\varepsilon = -1$, and the determinant condition requires $s = \pm 1$ in both cases. The spin points in the direction of $\mathbf{k}$ if $s = \varepsilon$ and in the opposite direction if $s = -\varepsilon$, cf. (A.6). In the absolute spacetime manifested by the cosmic microwave background and the expanding galaxy grid, it is quite natural to use the wave vector of the particle as a distinguished direction in its rest frame. The (anti-)particle moves with velocity $\mathbf{k}/\omega_k$ in the cosmic reference frame; particles with $s = 1$ are polarized in the direction of their propagation, and so are antiparticles (i.e. negative frequency modes, $\varepsilon = -1$) with $s = -1$. The direction of propagation is always $\mathbf{k}$, for particles and antiparticles alike, both species propagate forward in time, and the negative energies of antiparticles can be dealt with in second quantization, in the usual way, by anticommutation.

Four-spinors in different inertial frames connect by a similarity [27],

$$u = Su', \quad S = 2^{-1/2}\left(\sqrt{\gamma+1} + \boldsymbol{v}_0\boldsymbol{\alpha}\sqrt{\gamma-1}\right). \tag{2.8}$$

Here, the primed and unprimed frames relate by a boost with time component $t' = \gamma(t - \boldsymbol{v}\mathbf{x})$ and Lorentz factor $\gamma = (1-v^2)^{-1/2}$. The matrices $\boldsymbol{\alpha} = \alpha^k$ are defined in (2.6), $\boldsymbol{v}_0$ is the velocity unit vector, and $S^{-1} = S(-\boldsymbol{v}_0)$. We choose the primed frame as the rest frame of the (anti-)particle, which requires the identification $\omega_k = m\gamma$ and $\boldsymbol{v} = \mathbf{k}/\omega_k$ in (2.8). In the rest frame, the above separation ansatz simplifies to $\psi' = u'e^{i\varepsilon mt}$.

As pointed out after (2.7), the rest frame spinor for particles ($\varepsilon = 1$) of momentum $\mathbf{k}$ and spin s reads $u' = \sqrt{m/\omega_k}(\varphi'_{k,s}, 0)^{\mathrm{t}}$, where the 2-spinor is subjected to $(\boldsymbol{\sigma}\mathbf{k} - ks)\varphi'_{k,s} = 0$. By applying the similarity (2.8) with $\boldsymbol{v} = \mathbf{k}/\omega_k$, we find, in the cosmic reference frame,

$$u = \begin{pmatrix}\varphi_{k,s}\\ \chi_{k,s}\end{pmatrix} = \frac{1}{\sqrt{2\omega_k(\omega_k+m)}}\begin{pmatrix}(\omega_k+m)\varphi'_{k,s}\\ sk\varphi'_{k,s}\end{pmatrix}, \tag{2.9}$$

which solves $\hat{H}_0 u = \omega_k u$, cf. after (2.7). The same similarity is used for antiparticles ($\varepsilon = -1$) of momentum $\mathbf{k}$ and spin s, when transforming their rest frame spinor $u' = \sqrt{m/\omega_k}(0, \chi'_{k,s})^{\mathrm{t}}$ into the preferred cosmic frame,

$$u = \begin{pmatrix}\varphi_{k,s}\\ \chi_{k,s}\end{pmatrix} = \frac{1}{\sqrt{2\omega_k(\omega_k+m)}}\begin{pmatrix}-sk\chi'_{k,s}\\ (\omega_k+m)\chi'_{k,s}\end{pmatrix}. \tag{2.10}$$

This spinor solves $\hat{H}_0 u = -\omega_k u$. The primed rest frame 2-spinor satisfies $(\boldsymbol{\sigma}\mathbf{k} + ks)\chi'_{k,s} = 0$, cf. after (2.7), so that we may define, arbitrarily, $\chi'_{k,s} = \varphi'_{k,-s}$. The rest frame normalization of the 2-spinors will be taken as $\varphi'^{\dagger}_{k,s}\varphi'_{k,s} = \chi'^{\dagger}_{k,s}\chi'_{k,s} = 1$, cf. (A.9), so that $u^{\dagger}u = 1$, which gives a convenient normalization of the wave functions in the cosmic reference frame, cf. after (2.12). (Complex conjugation and transposition is indicated by a superscript †, for all quantities.)

We turn to the matrix elements of the 4-current, $j^{\kappa}_{mn} = -qc^2\bar{\psi}_n\gamma^{\kappa}\psi_m$, and decompose them into 2-spinors like in (2.4),

$$\rho_{mn} = q(\varphi_n^{\dagger}\varphi_m + \chi_n^{\dagger}\chi_m), \quad \mathbf{j}_{mn} = q(\chi_n^{\dagger}\sigma^k\varphi_m + \varphi_n^{\dagger}\sigma^k\chi_m), \tag{2.11}$$

where $\rho_{mn} = j^0_{mn}$ and $\mathbf{j}_{mn} = j^k_{mn}$. Here, we substitute the wave functions,

$$\psi_n = (\varphi_n, \chi_n)^{\mathrm{t}} = L^{-3/2}u_n \exp(i\varepsilon_n(\mathbf{k}_n\mathbf{x} - \omega_n t)), \tag{2.12}$$

where the multi-index $n = (\mathbf{k}_n, s_n, \varepsilon_n)$ indicates momentum, spin and frequency sign as defined after (2.7). We use the eigenfunctions $u_n = (\varphi_{k,s}, \chi_{k,s})^{\mathrm{t}}$ of $\hat{H}_0$ as stated in (2.9) and (2.10), with the helicity eigenfunctions $\varphi'_{k,s}$ and $\chi'_{k,s} = \varphi'_{k,-s}$ calculated in (A.9). Clearly, $(\varepsilon\omega - \varepsilon\boldsymbol{\alpha}\mathbf{k} - m\beta)u_n = 0$, and the dispersion relation follows by multiplying with $(\varepsilon\omega + \varepsilon\boldsymbol{\alpha}\mathbf{k} + m\beta)$, cf. the beginning of Appendix B.

For the remainder of this section, we will consider positive frequencies, $\varepsilon_n = 1$. We choose periodic boundary conditions on a box of size L, so that $\mathbf{k} = 2\pi\mathbf{n}/L$, with integer lattice points $\mathbf{n} \in Z^3$, which explains the normalization of the wave functions (2.12) by a factor of $L^{-3/2}$. We further define $\omega_{mn} := \omega_m - \omega_n$, and $\mathbf{k}_{mn} := \mathbf{k}_m - \mathbf{k}_n$, and factorize off the time dependence in (2.11), $\rho_{mn} = \tilde{\rho}_{mn}e^{-i\omega_{mn}t}$ and $\mathbf{j}_{mn} = \tilde{\mathbf{j}}_{mn}e^{-i\omega_{mn}t}$. The time separated matrix elements are assembled via (2.9–2.12),

$$\tilde{\rho}_{mn}(\mathbf{x}) = \frac{q}{L^3}\frac{(\omega_m+m)(\omega_n+m)+k_m k_n s_m s_n}{2\sqrt{\omega_m\omega_n(\omega_m+m)(\omega_n+m)}} \times \varphi_n'^{\dagger}\varphi_m' e^{i\mathbf{k}_{mn}\mathbf{x}},$$

$$\tilde{\mathbf{j}}_{mn}(\mathbf{x}) = \frac{q}{L^3}\frac{(\omega_m+m)k_n s_n+(\omega_n+m)k_m s_m}{2\sqrt{\omega_m\omega_n(\omega_m+m)(\omega_n+m)}} \times \varphi_n'^{\dagger}\boldsymbol{\sigma}\varphi_m' e^{i\mathbf{k}_{mn}\mathbf{x}}, \tag{2.13}$$

where $k_m = \sqrt{\omega_m^2 - m^2}$. The 2-spinors φ_m' are defined in (A.9).

The normalization of the wave functions (2.12), that is, of the spinors u_m in (2.9) and (2.10) and of the helicity functions φ_m' in (A.9), gives $\int_{L^3}\tilde{\rho}_{mn}d^3x = q\delta_{mn}$. We have put $\hbar = c = 1$ from (2.8) onwards. In (2.13) we have to restore these units in a way that $\tilde{\rho}_{mn} \sim qL^{-3}$ and $\tilde{\mathbf{j}}_{mn} \sim qcL^{-3}$. The spinors u_m and φ_m' as well as the Pauli matrices stay dimensionless. The continuity equation, $j^{\kappa}_{mn,\kappa} = 0$, reduces to $\omega_{mn}\tilde{\rho}_{mn}(0) = \mathbf{k}_{mn}\tilde{\mathbf{j}}_{mn}(0)$ (no summation here). This is consistent with the explicit formulas (2.13), by virtue of the hermiticity of the Pauli matrices and the eigenvalue equation $(\boldsymbol{\sigma}\mathbf{k}_m - k_m s_m)\varphi_m' = 0$ stated before (2.9).

The squared current matrix elements read, cf. (A.13),

$$|\tilde{\rho}_{mn}|^2 = \frac{q^2}{L^6}\frac{\omega_m\omega_n + m^2 + k_m k_n s_m s_n}{4\omega_m\omega_n k_m k_n} \times (k_m k_n + s_m s_n \mathbf{k}_m\mathbf{k}_n),$$

$$\left|\tilde{\mathbf{j}}_{mn}\right|^2 = \frac{q^2}{L^6}\frac{\omega_m\omega_n - m^2 + k_m k_n s_m s_n}{4\omega_m\omega_n k_m k_n} \times (3k_m k_n - s_m s_n \mathbf{k}_m\mathbf{k}_n). \tag{2.14}$$

They simplify if summed over the "final" spin s_n,

$$\sum_{s_n=\pm1}|\tilde{\rho}_{mn}|^2 = \frac{q^2}{L^6}\frac{\omega_m\omega_n + m^2 + \mathbf{k}_m\mathbf{k}_n}{2\omega_m\omega_n},$$

$$\sum_{s_n=\pm1}\left|\tilde{\mathbf{j}}_{mn}\right|^2 = \frac{q^2}{L^6}\frac{3\omega_m\omega_n - 3m^2 - \mathbf{k}_m\mathbf{k}_n}{2\omega_m\omega_n}. \tag{2.15}$$

The spin variable s_m does not show here any more, so that an average over the "initial" spin need not be considered. In the diagonal, we find q^2/L^6 and $q^2 v_m^2/L^6$, respectively.

3 Ionization of hydrogenic ground states by superluminal quanta

We explain the second quantization of the tachyon-electron interaction, that is, of the Proca field with negative mass-square, minimally coupled to Dirac spinors. We derive the transversal and longitudinal transition matrices and calculate the tachyonic ionization cross-sections of hydrogen-like ground states. These cross-sections will be scrutinized in some detail, e.g., for ultra-relativistic electronic ejection energies. We study the Dirac equation in an external electromagnetic potential, and treat the tachyon field as perturbation in linear order,

$$i\hbar\psi_{,t} = (H^{em} + H^{int})\psi, \quad H^{int} := -q(\alpha^k A_k + c^{-1}A_0),$$
$$H^{em} := -\hbar c i\alpha^k\nabla_k^{em} + mc^2\beta - ec^{-1}A_0^{em}. \tag{3.1}$$

The notation, the sign conventions, and the representation of the Dirac matrices employed in the appendices are defined in Section 2. We introduce the Hamiltonian densities $H_\psi^{em} = \psi^\dagger H^{em}\psi$ and $H_\psi^{int} = \psi^\dagger H^{int}\psi$, the latter can be written as $H_\psi^{int} = -c^{-1}A_\kappa j^\kappa$, with the 4-current $j^0 = \rho = q\psi^\dagger\psi$ and $j^k = qc\psi^\dagger\alpha^k\psi$, cf. the beginning of Section 2. The perturbation theory is based on "free" wave functions satisfying the unperturbed Dirac equation, $i\hbar\psi_{,t} = H^{em}\psi$, in an electromagnetic potential. We will need the current functionals $\rho(\psi,\varphi) = q\varphi^\dagger\psi$ and $\mathbf{j}(\psi,\varphi) = qc\varphi^\dagger\alpha^k\psi$, and we note the continuity equation, $\rho_{,t}(\psi,\varphi) + \mathrm{div}\mathbf{j}(\psi,\varphi) = 0$, for arbitrary free wave solutions. We will assume the electromagnetic field in H^{em} as time independent, so that we can use the factorization $\psi_{ns\varepsilon} = u_{ns\varepsilon}e^{-i\varepsilon\omega_n t}$. Here, $\varepsilon = \pm1$ stands for positive/negative frequency solutions, and s is a (spin) degeneration index, cf. after (2.7) and (2.12); the eigenfrequencies ω_n are positive or at least bounded from below. In this way, we arrive at the time separated Dirac equation, $\hbar\omega_n u_{ns\varepsilon} = \varepsilon H^{em}u_{ns\varepsilon}$. The energy spectrum labeled by n may well be continuous; we will use box quantization and perform the continuum limit when feasible. These eigenfunctions, $u_{ns\varepsilon}$, of the free Dirac Hamiltonian have to be explicitly known, by means of a separate perturbation theory. In a Coulomb potential, they are available in closed form [27], but even then it is better to use a perturbative expansion in the electric fine structure constant from scratch, cf. Appendix B, and to avoid the awkward exact basis vectors. We will restrict to positive frequencies, $\varepsilon = 1$, for notational convenience, and drop the index ε in the wave functions and eigenvectors.

In the above current functionals, we separate off the time dependence, defining $\rho(\psi_{mr},\psi_{ns}) =: \rho_{mr,ns}e^{-i\omega_{mn}t}$ and $\mathbf{j}(\psi_{mr},\psi_{ns}) =: \mathbf{j}_{mr,ns}e^{-i\omega_{mn}t}$, where $\omega_{mn} = \omega_m - \omega_n$. The time separated continuity equation thus reads $i\omega_{mn}\rho_{mr,ns} = \mathrm{div}\mathbf{j}_{mr,ns}$, so that the eigenfunctions u_{ns} can be subjected to the normalization condition $\int\rho_{mr,ns}d^3x = q\delta_{mn}\delta_{rs}$. We consider a free wave solution (in a time independent electromagnetic potential), $\psi = \sum_{ns}b_{ns}u_{ns}e^{-i\omega_n t}$, with arbitrary complex amplitudes b_{ns}. The field energy is diagonal and can be expanded as $E_0 = \int H_\psi^{em}d^3x = \sum_{ns}\hbar\omega_n b_{ns}^* b_{ns}$. In the current functionals, we put $\varphi = \psi$ and expand accordingly,

$$\rho = \sum_{mr,ns}\rho_{mr,ns}b_{ns}^* b_{mr}e^{-i\omega_{mn}t},$$
$$\mathbf{j} = \sum_{mr,ns}\mathbf{j}_{mr,ns}b_{ns}^* b_{mr}e^{-i\omega_{mn}t}, \tag{3.2}$$

where $\rho_{mr,ns} = qu_{ns}^{\dagger}u_{mr}$ and $\mathbf{j}_{mr,ns} = qcu_{ns}^{\dagger}\alpha^k u_{mr}$ are the time separated current matrices. These eigenmode expansions are to be substituted into the interaction Hamiltonian, $H_{\psi}^{int} = -c^{-1}(A_0\rho + \mathbf{Aj})$, together with the Fourier series of the free tachyon field, cf. the beginning of Section 2 and [28,29],

$$\mathbf{A}(\mathbf{x},t) = L^{-3/2}\sum_{\mathbf{k}}\left(\hat{\mathbf{A}}(\mathbf{k})e^{i(\mathbf{kx}-\omega t)} + \text{c.c.}\right),$$
$$\hat{\mathbf{A}}(\mathbf{k}) = \sum_{\lambda=1}^{3}\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\hat{a}(\mathbf{k},\lambda), \tag{3.3}$$

with $\mathbf{k} = 2\pi\mathbf{n}/L$. The summation is over integer lattice points $\mathbf{n}$ in R^3, corresponding to periodic boundary conditions. The $\boldsymbol{\varepsilon}_{\mathbf{k},1}$ and $\boldsymbol{\varepsilon}_{\mathbf{k},2}$ are arbitrary real unit vectors (linear polarization vectors) orthogonal to $\boldsymbol{\varepsilon}_{\mathbf{k},3} := \mathbf{k}_0 = \mathbf{k}/k$, so that the $\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}$ constitute an orthonormal triad, and the $\hat{a}(\mathbf{k},\lambda)$ are arbitrary complex numbers. The amplitudes $\hat{\mathbf{A}}$ can arbitrarily be prescribed, the time component $A_0(\mathbf{x},t)$ of the potential is then determined by the Lorentz condition and the free field equations, and so is the tachyonic dispersion relation, $k^2 = \omega^2/c^2 + m_t^2$. We split the potential (3.3) into transversal/longitudinal components $\mathbf{A}^{T,L}(\mathbf{x},t)$, defined by the Fourier coefficients,

$$\hat{\mathbf{A}}^T(\mathbf{k}) := \sum_{\lambda=1,2}\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\hat{a}(\mathbf{k},\lambda), \quad \hat{A}_0^T = 0, \tag{3.4}$$

$$\hat{\mathbf{A}}^L(\mathbf{k}) := \mathbf{k}_0\hat{a}(\mathbf{k},3), \; \hat{A}_0^L(\mathbf{k}) = -c^2k\omega^{-1}\hat{a}(\mathbf{k},3). \tag{3.5}$$

This decomposition is unique, as there is no gauge freedom. We have also indicated the Fourier coefficients of the time component $A_0(\mathbf{x},t)$, defined like in (3.3); the transversal time component vanishes, and $\omega(k)$ solves the above dispersion relation. The energy densities of the transversal and longitudinal wave fields read,

$$\left\langle\rho_E^T\right\rangle = 2c^{-2}\sum_{\mathbf{k};\lambda=1,2}\omega^2\hat{a}(\mathbf{k},\lambda)\hat{a}^*(\mathbf{k},\lambda),$$
$$\left\langle\rho_E^L\right\rangle = -2m_t^2\sum_{\mathbf{k}}\hat{a}(\mathbf{k},3)\hat{a}^*(\mathbf{k},3), \tag{3.6}$$

which are time averages obtained from classical Lagrange formalism [29]. To set up the second quantization of these densities, we introduce rescaled Fourier coefficients,

$$\hat{a}(\mathbf{k},\lambda) = 2^{-1/2}c\hbar^{1/2}\omega^{-1/2}a_{\mathbf{k},\lambda},$$
$$\hat{a}(\mathbf{k},3) = 2^{-1/2}\hbar^{1/2}\omega^{1/2}m_t^{-1}a_{\mathbf{k},3}, \tag{3.7}$$

$\lambda = 1,2$, to arrive at the mode decomposition,

$$\left\langle\rho_E^T\right\rangle = \sum_{\mathbf{k};\lambda=1,2}\hbar\omega_k a_{\mathbf{k},\lambda}a_{\mathbf{k},\lambda}^*, \quad \left\langle\rho_E^L\right\rangle = -\sum_{\mathbf{k}}\hbar\omega_k a_{\mathbf{k},3}a_{\mathbf{k},3}^*, \tag{3.8}$$

which is taken as starting point for the occupation number representation. The Fourier coefficients $a_{\mathbf{k},\lambda}$ are interpreted as operators, and the complex conjugates $a_{\mathbf{k},\lambda}^*$ as their adjoints $a_{\mathbf{k},\lambda}^+$. We use commutation relations for the transversal degrees, $\lambda = 1,2$, and anticommutators for the longitudinal modes, to turn the longitudinal energy density into a positive definite operator. The basis vectors of the creation/annihilation operators $a_{\mathbf{k},\lambda}^{(+)}$ in occupation number representation can be found in [29]. The time averaged transversal Hamilton operator of the free tachyon field is thus $\langle\rho_E^T\rangle$ in (3.8), with the Fourier amplitudes $a_{\mathbf{k},\lambda}a_{\mathbf{k},\lambda}^*$ replaced by the operator products $a_{\mathbf{k},\lambda}^+a_{\mathbf{k},\lambda}$. The longitudinal energy operator is defined by the substitution $a_{\mathbf{k},3}a_{\mathbf{k},3}^* \to -a_{\mathbf{k},3}^+a_{\mathbf{k},3}$ in $\langle\rho_E^L\rangle$. To sum up, commutation relations, $[a_{\mathbf{k},\lambda}, a_{\mathbf{k}',\lambda'}^+] = \delta_{\mathbf{kk}'}\delta_{\lambda\lambda'}$, hold for the transversal superluminal modes, and anticommutator brackets, $[a_{\mathbf{k},3}, a_{\mathbf{k}',3}^+]_+ = \delta_{\mathbf{kk}'}$, apply to the longitudinal degree as well as to the subluminal spinorial modes, $[b_{mr}, b_{ns}^+]_+ = \delta_{mn}\delta_{rs}$, cf. before (3.2). The tachyonic operators $a_{\mathbf{k},\lambda}^{(+)}$ commute with the particle operators $b_{ns}^{(+)}$.

We turn to the interaction operator. In the 4-current (3.2), we replace the amplitudes $b_{ns}^*b_{mr}$ of the Dirac field by the fermionic operator products $b_{ns}^+b_{mr}$. We consider positive frequencies only; antiparticles can be dealt with analogously. The transversal and longitudinal components, $\mathbf{A}^{T,L}(\mathbf{x},t)$ and $A_0^L(\mathbf{x},t)$, of the tachyon field, cf. (3.3–3.5), are turned into operators by replacing the rescaled Fourier coefficients $a_{\mathbf{k},\lambda}^{(*)}$ in (3.7) by the bosonic/fermionic $a_{\mathbf{k},\lambda}^{(+)}$, as done in the energy densities (3.8). The interaction operator, $H_{int}^T + H_{int}^L$, is obtained by substituting the tachyonic operator fields $\mathbf{A}^{T,L}$ and A_0^L as well as the operator current (3.2) into the interaction functional H_{ψ}^{int}, cf. after (3.2),

$$H_{int}^T = -c^{-1}\mathbf{A}^T\mathbf{j}, \quad H_{int}^L = -c^{-1}\left(A_0^L\rho + \mathbf{A}^L\mathbf{j}\right). \tag{3.9}$$

First, we settle the transversal interaction $\int H_{int}^T d^3x$, keeping the linear polarization λ fixed (that is, no summation over λ in (3.4)). The transition matrix elements for absorption and emission can readily be identified as,

$$\left\langle T_{abs/em}^T\right\rangle = -\frac{\hbar^{1/2}}{\sqrt{2}\omega_k^{1/2}L^{3/2}}\int\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\,\mathbf{j}_{mr,ns}\,e^{\pm i\mathbf{kx}}d^3x, \tag{3.10}$$

where $\mathbf{j}_{mr,ns}$ are the matrix elements of the time separated spinor current stated after (3.2). This corresponds to the absorption or emission of a single transversal tachyon, cf. [29,30] for details, where we studied interaction with a non-relativistic scalar particle in a Coulomb potential. The T-matrix for the interaction of tachyons with a Klein-Gordon particle was derived in [28]. We will skip the derivation of the transition rates here, as it is almost identical to the derivation given in these references for non-relativistic and spinless matter currents minimally coupled to the Proca field. The tachyonic wave vector $\mathbf{k}$ relates to the tachyonic frequency ω_k by the dispersion relation stated before (3.4). The matrices $\langle T_{abs}^T\rangle$ and $\langle T_{em}^T\rangle$ in (3.10) only differ by a sign change in the exponential; the upper sign always refers to absorption. The initial electronic state is indicated by a subscript m and the final state by n, so that a positive $\omega_{mn} := \omega_m - \omega_n$ stands for emission. The initial and final spin indices are denoted by r and s, respectively. The transition rate for transversally induced absorption and emission in a given linear polarization λ (where $\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}$ can be chosen quite arbitrarily, cf. after (3.3)) is obtained by a standard procedure [31],

$$dw_{abs/em}^{T,ind} \sim \frac{1}{8\pi^2}\frac{k}{\hbar c^2}\frac{1}{e^{\beta\hbar\omega}-1} \times \left|\int \boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\,\mathbf{j}_{mr,ns}\,e^{\pm i\mathbf{kx}}d^3x\right|^2 d\boldsymbol{\Omega}. \quad (3.11)$$

The tachyonic frequency ω (as well as $k(\omega)$) is taken at $|\omega_{mn}|$, and the solid angle element, $d\boldsymbol{\Omega} = \sin\theta d\theta d\varphi$, is centered at the tachyonic wave vector $\mathbf{k}$. The emission rate also applies to spontaneous radiation, if the $(e^{\beta\hbar\omega_k}-1)^{-1}$-factor (averaged occupation number) is dropped,

$$dw_{em}^{T,sp} \sim (e^{\beta\hbar\omega}-1)dw_{em}^{T,ind} =: A_{mr,ns}^{T}(\mathbf{k},\lambda)d\boldsymbol{\Omega}. \quad (3.12)$$

The spontaneous transversal emission rate is temperature independent, unaffected by the tachyonic heat bath, in contrast to the longitudinal emission discussed below. More importantly, the spontaneous emission rate is time symmetric, as manifested in the symmetry $A_{mr,ns}^{T}(\mathbf{k},\lambda) = A_{ns,mr}^{T}(-\mathbf{k},\lambda)$ of the Einstein coefficients, so that (3.12) also applies to the absorption of incident tachyons. The rates for unpolarized transversal radiation are obtained by replacing $\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\,\mathbf{j}_{mr,ns}$ in (3.11) by the transversal current matrix, $\mathbf{j}_{mr,ns}^{T} = \mathbf{j}_{mr,ns} - \mathbf{k}_0(\mathbf{k}_0\mathbf{j}_{mr,ns})$, cf. (3.3).

The longitudinal component of the interaction, H_{int}^{L} in (3.9), can be dealt with analogously. We split this Hamiltonian into $H_{int}^{L(1)} = -c^{-1}\mathbf{A}^L\mathbf{j}$ and $H_{int}^{L(2)} = -c^{-1}A_0^L\rho$, so that the respective T-matrix components read,

$$\left\langle T_{abs/em}^{L(1)}\right\rangle = -\frac{\hbar^{3/2}\omega_k^{1/2}}{\sqrt{2}m_t c^2 L^{3/2}}\int \mathbf{k}_0\,\mathbf{j}_{mr,ns}e^{\pm i\mathbf{kx}}d^3x,$$

$$\left\langle T_{abs/em}^{L(2)}\right\rangle = \frac{\hbar^{3/2}k}{\sqrt{2}m_t\omega_k^{1/2}L^{3/2}}\int \rho_{mr,ns}e^{\pm i\mathbf{kx}}d^3x. \quad (3.13)$$

We have here restored the mass unit, $m_t \to m_t c/\hbar$, and $\mathbf{k}_0 = \mathbf{k}/k$ is the tachyonic unit wave vector. Employing the continuity equation as stated before (3.2), we can express the longitudinal transition matrix , $\langle T^{L(1)}\rangle + \langle T^{L(2)}\rangle$, by the charge density only,

$$\left\langle T_{abs/em}^{L}\right\rangle = \frac{m_t c^2}{\sqrt{2}\hbar^{1/2}\omega_k^{1/2}kL^{3/2}}\int \rho_{mr,ns}e^{\pm i\mathbf{kx}}d^3x. \quad (3.14)$$

Here, we used energy conservation, $\omega_k = \mp\omega_{mn}$, as well as the tachyonic dispersion relation, $k^2 = \omega_k^2/c^2 + (m_t c/\hbar)^2$. The longitudinal transition rates read accordingly,

$$dw_{abs/em}^{L,ind} \sim \frac{1}{8\pi^2}\frac{m_t^2c^2}{\hbar^3 k}\frac{1}{e^{\beta\hbar\omega}+1}\left|\int \rho_{mr,ns}e^{\pm i\mathbf{kx}}d^3x\right|^2 d\boldsymbol{\Omega}. \quad (3.15)$$

The lower minus sign refers to emission. The subscript mr denotes the initial electronic state, for absorption and emission alike, and $\omega = |\omega_{mn}|$. We write the total emission rate as $dw_{em}^{L} = dw_{em,T=0}^{L,sp} - dw_{em}^{L,ind}$, with $dw_{em}^{L,ind}$ in (3.15) and $dw_{em,T=0}^{L,sp} := (e^{\beta\hbar\omega}+1)dw_{em}^{L,ind}$, the latter being the spontaneous transition rate in the zero temperature limit. At finite temperature, the spontaneous emission rate is identified as $dw_{em}^{L,sp} = dw_{em,T=0}^{L,sp} - 2dw_{em}^{L,ind}$, so that the total emission, $dw_{em}^{L} = dw_{em}^{L,ind} + dw_{em}^{L,sp}$, is properly accounted for. Hence,

$$dw_{em}^{L,sp} \sim \tanh(\beta\hbar\omega/2)dw_{em,T=0}^{L,sp} =: A_{mr,ns}^{L}(\mathbf{k})d\boldsymbol{\Omega}. \quad (3.16)$$

The symmetry $A_{mr,ns}^{L}(\mathbf{k}) = A_{ns,mr}^{L}(-\mathbf{k})$ of the A-coefficients also extends to longitudinal radiation. The longitudinal spontaneous emission is thus temperature dependent and vanishes in the high-temperature limit.

We scrutinize in greater detail the tachyonic photoeffect, the ejection of a bound electron into the continuum by an incoming tachyon. We start with the absorption rates [31],

$$w_{abs}^{T,L} \sim \frac{n_{\mathbf{k}}}{t\hbar^2}\sum_{\mathbf{k}_n}\left|\left\langle T_{abs}^{T,L}\right\rangle\right|^2\left|\int_{-t/2}^{t/2}e^{-i(\omega_{mn}+\omega_k)t}dt\right|^2$$

$$\sim \frac{2\pi n_{\mathbf{k}}}{\hbar^2c^2}\frac{L^3}{(2\pi)^3}\int d\boldsymbol{\Omega}\int_{mc^2/\hbar}^{\infty}\left|\left\langle T_{abs}^{T,L}\right\rangle\right|^2 \times \delta(\omega_m - \omega_n + \omega_k)k_n\omega_n d\omega_n, \quad (3.17)$$

where we have replaced the summation over the electronic wave vectors by the continuum limit, $L^3(2\pi)^{-3}\int d\mathbf{k}_n$, and used the subluminal dispersion relation, $k_n^2 = \omega_n^2/c^2 - (mc/\hbar)^2$, to obtain $d\mathbf{k}_n = c^{-2}k_n\omega_n d\omega_n d\boldsymbol{\Omega}$. The angle increment thus refers to the electronic wave vector in this bound-free transition, in contrast to the transition rates stated above, where $d\boldsymbol{\Omega} = \sin\theta d\theta d\varphi$ is the solid angle element of the tachyonic wave vector. The occupation numbers $n_{\mathbf{k}}$ label the incident tachyon flux.

The absorption rates are readily assembled by means of the transition matrices $\langle T_{abs}^{T,L}\rangle$ in (3.10) and (3.14),

$$dw_{abs}^{T} \sim \frac{n_{\mathbf{k}}}{8\pi^2}\frac{k_n\omega_n}{\hbar c^2\omega_k}\left|J_{mr,ns}^{T}\right|^2 d\boldsymbol{\Omega},$$

$$dw_{abs}^{L} \sim \frac{n_{\mathbf{k}}}{8\pi^2}\frac{m_t^2c^2k_n\omega_n}{\hbar^3k^2\omega_k}\left|J_{mr,ns}^{L}\right|^2 d\boldsymbol{\Omega}, \quad (3.18)$$

where $J_{mr,ns}^{T}$ and $J_{mr,ns}^{L}$ denote the integrals in (3.10) and (3.14), both with a plus sign in the exponent, cf. (B.8). Energy conservation, $\omega_n = \omega_m + \omega_k$, is implied in (3.18). We will focus on ground state ionization, so that ω_m stands for the electronic ground state energy, and we will write p and ω_p for the electronic wave vector and frequency of the final state (the ejected free electron), instead of k_n and ω_n. We parametrize the energy of the free electron with the Lorentz factor, $\hbar\omega_p = mc^2\gamma$, so that we have to add the rest mass to the non-relativistic ground state energy, $\hbar\omega_m \approx mc^2 - E_0$, where $E_0 = mc^2\alpha_Z^2/2$. Hence, $\hbar\omega_p \approx mc^2 - E_0 + \hbar\omega_k$. The energy of the ionizing tachyon can likewise be parametrized by its speed, $\hbar\omega_k = m_t c^2\gamma_t$, where $\gamma_t = (\upsilon_t^2/c^2 - 1)^{-1/2}$ is the tachyonic Lorentz factor, but it is more efficient to parametrize ω_k with the electronic Lorentz factor, making use of energy conservation. When calculating the matrix elements $J^{T,L}_{mr,ns}$, in Appendix B, we will need the α_Z-expansion of the electronic scattering state [27], which requires $\hbar\omega_k \gg E_0$, cf. (B.7). Therefore, we can drop the ionization energy E_0 in the energy conservation, so that $\hbar\omega_k \approx mc^2(\gamma - 1)$. (At the ionization threshold, $\hbar\omega_k \approx E_0$, we would have to use the dipole approximation with exact but non-relativistic wave functions. This limit is of special interest when applied to Rydberg states [32,33]. At the ionization threshold of highly excited states of order $n \sim 10^4$, the longitudinal cross-section starts to compete with photoionization [29], in recombination even at lower levels.) The wave vectors can be parametrized with γ via the dispersion relations $k^2 = \omega_k^2 + m_t^2$ and $p^2 = \omega_p^2 - m^2$. When restoring units, k and p denote wave vectors, rather than momenta.

The current density of the incoming tachyons is $\upsilon_{gr} n_{\mathbf{k}}/L^3$, where $\upsilon_{gr} = c^2 k/\omega_k$, so that the transversal and longitudinal cross-sections relate to the respective absorption rates (3.18) as $d\sigma^{T,L} = L^3\omega_k(c^2 k n_{\mathbf{k}})^{-1} dw^{T,L}_{abs}$. Hence,

$$d\sigma^T = \frac{L^3\omega_p p}{8\pi^2 c^4\hbar k}\left\langle J^T\right\rangle^2 d\mathbf{\Omega}, \quad d\sigma^L = \frac{L^3 m_t^2\omega_p p}{8\pi^2\hbar^3 k^3}\left\langle J^L\right\rangle^2 d\mathbf{\Omega}. \tag{3.19}$$

Here, we replaced $|J^{T,L}_{mr,ns}|^2$ by $\langle J^{T,L}\rangle^2 := (1/2)\sum_{r,s=\pm 1}|J^{T,L}_{mr,ns}|^2$, averaging the cross-sections over the initial spins and summing over the final ones. The averaged matrix elements $\langle J^{T,L}\rangle^2$ are calculated in Appendix B, in polar parametrization, cf. (B.30) and (B.31). We use the asymptotic wave vector $\mathbf{k}$ of the ionizing tachyon as polar axis, $\mathbf{pk} = pk\cos\theta$. The transversal linear polarization vector defining the azimuthal parametrization of the electronic momentum can be arbitrarily chosen, $\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\mathbf{p} = p\sin\theta\cos\varphi$. The results of Appendix B are summarized in (3.20–3.25). We find the transversal cross-section for relativistic ejection energies as, cf. (B.32),

$$d\sigma^T = \frac{16\alpha_q\alpha_Z^5\hbar^2}{\eta\delta^2c^2m^2}\frac{\sqrt{\gamma^2-1}(\gamma+1)}{(\gamma-1)^4a^4(\theta)}\varSigma^T d\mathbf{\Omega}, \tag{3.20}$$

$$\varSigma^T := \frac{1}{4(\gamma-1)^2}\frac{m_t^4}{m^4}a(\theta)+\sin^2\theta\left[\frac{1}{4}\left((\gamma-1)^2+\frac{m_t^2}{m^2}\right)a(\theta) -\frac{1}{2}\frac{m_t^2}{m^2}\eta^2+\delta\cos^2\varphi\left(2\delta-\frac{1}{2}(\gamma-1)a(\theta)\right)\right], \tag{3.21}$$

and the longitudinal differential cross-section reads, cf. (B.33),

$$d\sigma^L = \frac{4\alpha_q\alpha_Z^5\hbar^2m_t^2}{\eta\delta^2c^2m^4}\frac{\sqrt{\gamma^2-1}(\gamma+1)}{(\gamma-1)^6a^4(\theta)}\varSigma^L d\mathbf{\Omega}, \tag{3.22}$$

$$\varSigma^L := 4a(\theta)+\sin^2\theta\left[2\frac{m_t^2}{m^2}-8\gamma+(\gamma^2-1)a(\theta)\right]. \tag{3.23}$$

In (3.20–3.23), we have introduced the shortcuts,

$$\eta^2 := 1+\frac{1}{(\gamma-1)^2}\frac{m_t^2}{m^2}, \quad \delta := 1-\frac{1}{2(\gamma-1)}\frac{m_t^2}{m^2}, \tag{3.24}$$

$$a(\theta) := 2\gamma\chi\left(1-\frac{\upsilon}{c}\frac{\eta}{\chi}\cos\theta\right), \quad \chi := 1+\frac{1}{2\gamma(\gamma-1)}\frac{m_t^2}{m^2}. \tag{3.25}$$

The derivation of these cross-sections is given in Appendix B. The dimensionless fine structure constant, $\alpha_q = q^2/(4\pi\hbar c)$, is defined after (2.1), and $\alpha_Z \approx Z/137$, cf. after (B.2). The angular dependent factor $a(\theta)$ in (3.25) enters in fourth power in the denominators of the cross-sections. It cannot attain zero for any scattering angle, since $\upsilon\eta/(c\chi) < 1$ (where υ is the speed of the ejected electron, $\upsilon/c = (\gamma^2-1)^{1/2}/\gamma$). This inequality is satisfied for all γ, apart from the double zero $\gamma = 1 + m_t^2/(2m^2)$, which corresponds to $\upsilon/c \approx m_t/m$. In the calculation of the matrix elements in (3.19), we use an approximation of the electronic scattering state which requires $\upsilon/c \gg m_t/m$ and even the stronger condition that the ionization energy is negligible compared to the kinetic energy of the ejected electron, cf. the remarks on energy conservation following (3.18) and the Born approximation discussed below, after (3.31). Thus the factor $a(\theta)$ does not give rise to a singularity or a diverging total cross-section.

The total cross-sections are obtained by performing the solid angle integration in (3.20) and (3.22), which can be reduced to the integrals listed in (B.34) and (B.35). We find the total transversal section as,

$$\begin{aligned}\sigma^T &= \frac{\pi\alpha_q\alpha_Z^5\hbar^2\upsilon}{3\eta^3\delta^6c^3m^2}\frac{\gamma}{(\gamma-1)^4}S^T,\\ S^T &:= 2(3\gamma^2-2\gamma+4)-3\left(\gamma-2+\frac{3}{2}\frac{m_t^2}{m^2}\frac{1}{\gamma-1}\right)\\ &\quad\times\frac{\delta^4c}{\eta\upsilon\gamma}\log\frac{1+\eta\upsilon/(\chi c)}{1-\eta\upsilon/(\chi c)}\\ &\quad-2\frac{m_t^2}{m^2}\frac{5\gamma^3-13\gamma^2+9\gamma-9}{(\gamma-1)^2}\\ &\quad+\frac{1}{2}\frac{m_t^4}{m^4}\frac{15\gamma^3-45\gamma^2+23\gamma-25}{(\gamma-1)^3}\\ &\quad+\frac{m_t^6}{m^6}\frac{12\gamma^2-5\gamma+1}{(\gamma-1)^4}+\frac{3}{8}\frac{m_t^8}{m^8}\frac{11\gamma+5}{(\gamma-1)^5}.\end{aligned} \tag{3.26}$$

The electromagnetic cross-section σ^{ph} is recovered from σ^T, if we put $m_t = 0$ (which implies $\eta = \delta = \chi = 1$), and replace α_q with the electric fine structure constant, $\alpha_e \approx 1/137$, cf. [27]. The total longitudinal cross-section reads,

$$\sigma^L = \frac{2\pi\alpha_q\alpha_Z^5\hbar^2 m_t^2 \upsilon^3}{3\eta^3\delta^6 c^5 m^4}\frac{\gamma^3}{(\gamma-1)^7}S^L,$$

$$S^L{:} = 7\gamma - \frac{3}{2}\frac{\delta^4 c}{\eta\upsilon\gamma}\log\frac{1+\eta\upsilon/(\chi c)}{1-\eta\upsilon/(\chi c)} - \frac{1}{2}\frac{m_t^2}{m^2}\frac{2\gamma^2-21\gamma+11}{(\gamma-1)^2} + \frac{1}{4}\frac{m_t^4}{m^4}\frac{3\gamma^2-\gamma+22}{(\gamma-1)^3} + \frac{3}{8}\frac{m_t^6}{m^6}\frac{1}{(\gamma-1)^3}. \quad (3.27)$$

The recombination cross-sections are obtained by balancing emission and absorption rates, $\sigma^T_{rec} = 2(k/p)^2\sigma^T$ and $\sigma^L_{rec} = (k/p)^2\sigma^L$, reflecting the symmetry of the Einstein coefficients. The factor of two is the weight of the transversal degrees. These cross-sections refer to electron capture in the empty K-shell, irrespectively of the spin. (The ionization cross-sections $\sigma^{T,L}$ assume a single electron in the K-shell.) As pointed out after (3.18), the ratio of the tachyonic and electronic wave vectors can be parametrized with the electronic Lorentz factor,

$$\frac{k^2}{p^2} \approx \frac{\gamma-1}{\gamma+1} + \frac{m_t^2}{m^2}\frac{1}{\gamma^2-1}, \quad (3.28)$$

where we have neglected the ionization energy. In strong contrast to the ionization threshold, the factor k/p cannot get large in the high-energy regime and augment the likelihood of recombination by tachyon emission.

We study limit cases of the above cross-sections. In the ultra-relativistic limit, $\gamma \gg 1$, we can approximate $S^T \sim 6\gamma^2$ and $S^L \sim 7\gamma$ in (3.26) and (3.27), so that the total cross-sections simplify to

$$\sigma^T \sim \frac{2\pi\alpha_q\alpha_Z^5\hbar^2}{c^2m^2\gamma}, \quad \sigma^L \sim \frac{14\pi\alpha_q\alpha_Z^5\hbar^2 m_t^2}{3c^2m^4\gamma^3}, \quad \frac{\sigma^L}{\sigma^T} \sim \frac{7}{3}\frac{m_t^2}{m^2}\frac{1}{\gamma^2}, \quad (3.29)$$

which are the leading orders in an $1/\gamma$-expansion. We note $m_t/m \approx 1/238$, and $\sigma^T/\sigma^{ph} \sim \alpha_q/\alpha_e \approx 1.4 \times 10^{-11}$. The ratio σ^L/σ^T also shows in the quotients of the differential sections, so that the transversal radiation largely overpowers the longitudinal emission in this limit. The angular extrema of the differential sections (3.20) and (3.22), that is, of $d\sigma^{T,L}/d\theta d\varphi$, solve

$$a(\theta) d\left(\sin\theta\,\Sigma^{T,L}\right)/d\theta = 4a'(\theta)\sin\theta\,\Sigma^{T,L}. \quad (3.30)$$

In the ultra-relativistic limit, we use the ansatz $\cos\theta = 1 - x^2/(2\gamma^2) + ...$, so that $a(\theta) \sim (1+x^2)/\gamma$ and $a'(\theta) \sim 2x$, in leading order in $1/\gamma$. The transversal maximum is just $x \approx 1$, corresponding to a scattering angle of $\theta \approx 1/\gamma$. The longitudinal peak is found as the positive zero of $3x^6 - 20x^4 + 37x^2 - 4 = 0$, namely $x \approx 0.3393$, so that $\theta \approx 0.34/\gamma$. The differential cross-sections greatly simplify for scattering angles of order $1/\gamma$, as we may approximate $\sin\theta \sim \theta$ and $a(\theta) \sim (1+\gamma^2\theta^2)/\gamma$ in (3.20–3.23). Hence, in the case of ultra-relativistic ejection energies and for scattering angles close to the maxima, the differential cross-sections simplify to,

$$d\sigma^T \sim \frac{4\alpha_q\alpha_Z^5\hbar^2}{c^2m^2}\frac{\gamma^3\theta^3 d\theta d\varphi}{(1+\gamma^2\theta^2)^3},$$

$$d\sigma^L \sim \frac{4\alpha_q\alpha_Z^5\hbar^2 m_t^2}{c^2m^4}\frac{(4-3\gamma^2\theta^2+\gamma^4\theta^4)\theta d\theta d\varphi}{\gamma(1+\gamma^2\theta^2)^4}, \quad (3.31)$$

which is the leading order of the $1/\gamma$-expansion, exhibiting marked peaks.

We turn to the non-relativistic limit of the total cross-sections (3.26) and (3.27), in Born approximation, and put $\gamma \approx 1 + \upsilon^2/(2c^2)$, but still assuming $\upsilon/c \gg m_t/m$. Even $\upsilon/c \gg \alpha_Z$ is required by the Born approximation in a Coulomb potential, tantamount to ejection energies much higher than the ionization threshold, cf. the remarks following (3.18) and (3.25). This suggests to expand $S^{T,L}$ in the small ratio $m_t c/m\upsilon$ (as well as in υ/c). We put $\delta \approx \chi \approx 1$, valid up to terms of $\mathrm{O}((m_t c/m\upsilon)^2)$. To the same accuracy, $\eta^2 \approx 1 + 4m_t^2c^4/(m^2\upsilon^4)$. The ratio $m_t c^2/m\upsilon^2$ need not be small, but $\eta\upsilon/c \ll 1$ holds safely. In this way, we find $S^T \sim 4S^L \sim 16\eta^2$; the first three terms in (3.26) and (3.27) contribute to that. Finally, we write $\eta \approx 2\varepsilon/(m\upsilon^2)$, where $\varepsilon^2 := (m\upsilon^2/2)^2 + m_t^2c^4$, to recover the non-relativistic Born approximation [29,30],

$$\sigma^T \sim \frac{2^7\pi}{3}\frac{\alpha_q\alpha_Z^5\hbar^2c^5}{\varepsilon m\upsilon^5}, \quad \sigma^L \sim \frac{2^9\pi}{3}\frac{\alpha_q\alpha_Z^5\hbar^2 m_t^2 c^9}{\varepsilon m^3\upsilon^9}. \quad (3.32)$$

The recombination cross-sections relate to (3.32) as stated after (3.27); the non-relativistic limit of (3.28) reads $k/p \approx \varepsilon/(m\upsilon c)$.

The non-relativistic limit of the differential sections (3.20–3.23) is easily found, by making use of the expansions outlined after (3.31),

$$\frac{d\sigma^T}{d\theta d\varphi} = \frac{2^5\alpha_q\alpha_Z^5\hbar^2c^5}{\varepsilon m\upsilon^5}\sin^3\theta\cos^2\varphi \times \left(1 + \frac{8\varepsilon}{m\upsilon c}\cos\theta + \mathrm{O}\left(\frac{\upsilon^2}{c^2}, \frac{m_t^2c^2}{m^2\upsilon^2}\right)\right), \quad (3.33)$$

$$\frac{d\sigma^L}{d\theta d\varphi} = \frac{2^7\alpha_q\alpha_Z^5\hbar^2 m_t^2 c^9}{\varepsilon m^3\upsilon^9}\sin\theta \times \left[\cos^2\theta\left(1 + \frac{8\varepsilon}{m\upsilon c}\cos\theta\right) - \frac{2\varepsilon}{m\upsilon c}\cos\theta + \mathrm{O}\right]. \quad (3.34)$$

The Born approximation is only valid for $\upsilon/c \gg m_t/m$. Therefore, we have included the ε-terms, that is, the first order correction in υ/c and $m_t c/(m\upsilon)$. When performing the angular integration, the ε-terms drop out, and we recover the total cross-sections (3.32). However, the ε-terms shift the angular extrema. The transversal section is peaked at θ^T_{max}, very close to the minimum of the longitudinal section at θ^L_{min}, and this minimum separates two longitudinal peaks located at $\theta^L_{max1,2}$, where

$$\theta^T_{max} \approx \frac{\pi}{2} - \frac{8}{3}\frac{\varepsilon}{m\upsilon c}, \quad \theta^L_{min} \approx \frac{\pi}{2} - \frac{\varepsilon}{m\upsilon c},$$

$$\theta^L_{max1,2} \approx \theta_{1,2} - \frac{11}{6\sqrt{3}}\frac{\varepsilon}{m\upsilon c}. \quad (3.35)$$

The angles $\theta_{1,2}$ defining the longitudinal maxima are the roots of $\sin\theta = 1/\sqrt{3}$, that is, $\theta_1 \approx 0.6155$ and $\theta_2 = \pi - \theta_1$. Thus the peaks of the longitudinal section (3.34) occur at scattering angles of 35.3° and 144.7° (without ε-shift) and have the same height. The ε-correction augments the peak of $d\sigma^L$ at θ^L_{max1} and attenuates the second peak at θ^L_{max2}. The height of the transversal peak and the longitudinal minimum are not affected by the ε-shift, in linear order at least. The total non-relativistic cross-sections (3.32) can, of course, be derived from non-relativistic scattering theory (Schrödinger equation in a Coulomb potential, minimally coupled to the Proca field [30]), we do not even have to take spin into account in the non-relativistic limit. The differential cross-sections (3.33) and (3.34) can also be recovered in this way, but without ε-correction. In [30], we actually guessed the relativistic first-order correction of the transversal section, relying on the electromagnetic counterpart, but we didn't attempt to find the longitudinal correction terms. The angular maxima cited in [30] differ from those in (3.35), as we there calculated the extrema of $d\sigma^{T,L}/d\mathbf{\Omega}$ instead of $d\sigma^{T,L}/d\theta d\varphi$. In any case, there is a clear separation of the transversal and longitudinal peaks, which can be used to distinct tachyons of different polarization, and to disentangle longitudinal and transversal radiation. In the ultra-relativistic limit, the transversal and the longitudinal section are both peaked at small scattering angles, but they cannot coalesce since $\theta^L_{max}/\theta^T_{max} \sim 0.34$, and the peaks become much more pronounced than in the non-relativistic limit.

4 Conclusion

The quantization of superluminal field theories has constantly been marred by the fact that there is no relativistically invariant way to distinct positive and negative frequency solutions outside the lightcone, a consequence of time inversions by Lorentz boosts. When attempting quantization, the result was either unitarity violation or non-invariant vacua [12–16]. Therefore, relativistic interactions of superluminal quanta with matter have never been worked out to an extend that they could be subjected to test. In fact, tachyons have not been detected so far, and one may ask why. There is the possibility that superluminal signals just don't exist, the vacuum speed of light being the definitive upper bound. In an open universe, however, this is not a particularly appealing perspective. There is another explanation, which we worked out quite quantitatively in this paper. The interaction of superluminal radiation with matter is very small, the quotient of tachyonic and electric fine structure constants being $\alpha_q/\alpha_e \approx 1.4 \times 10^{-11}$, and therefore superluminal quanta are just hard to detect. There have been searches for tachyons, which were assumed to be electrically charged, emitting Cherenkov radiation in vacuum, and bubble-chamber events were reanalyzed in search of negative mass-squares of neutral tachyons inferred from energy-momentum conservation [5,6,9–11]. Apart from that, tachyonic quanta should have been detected over the years, accidentally, despite of their tiny interaction with matter. The most likely reason as to why this has not happened is this: due to our contemporary relativistic spacetime conception, we are obliged to systematically ignore them.

In strong contrast to (sub-)luminal wave propagation, there is no retarded propagator supported outside the lightcone, only a time symmetric Green function generating half-retarded half-advanced wave fields. To achieve fully retarded wave propagation, an absorber is needed that turns the advanced component into the missing half of the retarded field [34,35]. The absorber field, a local manifestation of the absolute cosmic spacetime, affects the energy balance, the mentioned search for missing negative mass-squares [6] fails since the energy radiated is drained from the absorber [24,28]. There is no vacuum Cherenkov radiation either [3,5], as the radiating sources are subluminal.

In Section 3, we have quantized the free superluminal radiation field and the interaction Hamiltonian. There are three major deviations from the standard quantization procedure of subluminal field theories. Two of them are technical, the third involves the absolute cosmic time order. First, it is the time averaged energy density, rather than the classical Hamiltonian, that is taken as starting point for the operator interpretation. Second, Fermi statistics is employed for the longitudinal modes of an integer spin field, and third, the vacuum state is defined with respect to the cosmic reference frame [24]. As mentioned in the Introduction, Fermi statistics for integer spin fields was already employed by Feinberg [4]. However, he had to restrict his theory to a scalar field, as there is no relativistically invariant unitary representation of higher spin with negative mass-square. Lorentz boosts mix not only positive and negative frequency solutions but also the longitudinal and transversal components of vector fields [30]. In the absolute cosmic spacetime, we are not hampered by the lack of unitary vector representations of the Lorentz group with negative mass-square, since the S-matrix is safely unitary in the rest frames of comoving observers, as exemplified in Section 3, where we calculated the transition matrix for tachyonic bound-bound and bound-free transitions in a Coulomb potential. The absolute cosmic time order provides a way to unambiguously identify advanced and retarded wave fields, positive and negative frequency solutions, particles and antiparticles, and to define a stable vacuum state and a positive definite energy operator, all this is key to quantization [29].

Differential cross-sections are perhaps the most practical means to disentangle transversal and longitudinal radiation, as demonstrated here with ionization. The polarization of the ionizing radiation affects the angular maxima, the peaks in the transversal and longitudinal cross-sections occur at different scattering angles. One may expect that tachyonic Compton scattering can also be used to discern longitudinal quanta from transversal tachyons and photons. There should be a transversal and longitudinal tachyonic counterpart to the Klein-Nishina formula, pertinent to the acceleration of the electron by the incoming tachyonic wave field triggering electromagnetic radiation. The tachyonic Thomson cross-section, the non-relativistic classical limit, was already derived in [24], but a quantum mechanical version is still lacking, especially if the energies of the incident tachyonic X-rays are close to the tachyon mass. Another interesting cross-section to be scrutinized in search of longitudinal radiation is the conversion of tachyonic γ-rays [36] into electron-positron pairs. Pair production by tachyons has not been studied in any limit and context as yet, for instance, in a Coulomb potential or strong magnetic field. In the ultra-relativistic limit, the cross-section for the conversion of transversal tachyonic γ-rays is presumably just the Bethe-Heitler formula rescaled by the ratio α_q/α_e of the fine structure constants. Other mechanisms for the detection of longitudinal radiation modes have been suggested, pertaining to a finite photon mass (positive mass-square), such as a capacitor in a perfectly conducting shell, impenetrable for transversal waves [19]. This line of reasoning, focused on macroscopic current distributions, is unlikely to be applicable to tachyons. At least, there is no obvious tachyonic counterpart to a perfectly conducting shell or the skin depth of a conductor, or even to a macroscopic charge density, due to averaging effects caused by periodic sign changes of the tachyon potential [20].

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Appendix A: Interaction of tachyons with slow electrons: matrix elements in the low-energy limit

We use the notation of Section 2. In the non-relativistic limit, the Dirac equation can be replaced by the Pauli equation,

$$\frac{1}{i}\nabla_0^A\psi = \frac{\hbar}{2m}\left(\nabla_k^A\nabla^{Ak} + \sigma_i\varepsilon^{ikl}\left(\tilde{e}A_{l,k}^{em} + \tilde{q}A_{l,k}\right)\right)\psi, \tag{A.1}$$

where ψ is a 2-spinor field. The derivatives ∇_μ^A are defined in (2.1), and the σ_i are the usual Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.2}$$

The indices of σ^k are lowered with δ_{ik}, and we will occasionally write $\boldsymbol{\sigma}$ for this 3-vector. We note $\sigma^m\sigma^n = i\varepsilon^{mnk}\sigma_k + \delta^{mn}$ as well as $\varepsilon_{kij}\sigma^i\sigma^j = 2i\sigma_k$, where ε^{ijk} is the totally antisymmetric Levi-Civita symbol, so that $\varepsilon^{ijk}\varepsilon_{mnk} = \delta_m^i\delta_n^j - \delta_n^i\delta_m^j$. A possible Lagrangian for the wave equation (A.1) is $L = L_t + L_s$, where

$$L_t = \frac{i\hbar}{2}\left(\psi^\dagger\nabla_0^A\psi - \left(\nabla_0^{A*}\psi^\dagger\right)\psi\right),$$
$$L_s = -\frac{\hbar^2}{2m}\left(\left(\nabla_i^{A*}\psi^\dagger\sigma^i\right)\sigma^k\nabla_k^A\psi\right). \tag{A.3}$$

Here, $\psi^\dagger$ indicates transposition and complex conjugation of the 2-spinor. This Lagrangian is to be identified with L_{int} when added to the Proca Lagrangian L_P, cf. the beginning of Section 2. We may replace the component containing the spatial derivatives by

$$L_s = -\frac{\hbar^2}{2m}\left((\nabla^{Ak*}\psi^\dagger)\nabla_k^A\psi - \varepsilon^{ikl}(\tilde{e}A_i^{em} + \tilde{q}A_i)\partial_k(\psi^\dagger\sigma_l\psi)\right), \tag{A.4}$$

which coincides with L_s in (A.3) up to a divergence. Here, $A_i^{(em)}\partial_k$ can be replaced by $A_{k,i}^{(em)}$, again up to a divergence, so that the Pauli equation (A.1) easily follows, but the current should be derived from (A.3) or (A.4), via $j^\mu = c\partial L/\partial A_\mu$, to reproduce the Dirac current in the non-relativistic limit, cf. after (A.6). In so doing, we find the charge density, $\rho = j^0 = q\psi^\dagger\psi$, and the 3-current,

$$j^i = \frac{i\hbar q}{2m}\left(\left(\nabla^{Ai*}\psi^\dagger\right)\psi - \psi^\dagger\nabla^{Ai}\psi\right) + \frac{\hbar q}{2m}\varepsilon^{ikl}\partial_k\left(\psi^\dagger\sigma_l\psi\right). \tag{A.5}$$

The signs have been chosen in a way that j^μ can be identified with the current in the Proca equation as stated at the beginning of Section 2.

The non-relativistic Hamiltonian is identified by writing the Pauli equation (A.1) as $i\hbar\psi_{,t} = H\psi$,

$$-H = \frac{\hbar^2}{2m}\nabla_k^A\nabla^{Ak} + \frac{1}{mc}\mathbf{S}\cdot(e\mathbf{B}^{em} + q\mathbf{B}) + \frac{e}{c}A_0^{em} + \frac{q}{c}A_0, \tag{A.6}$$

where $\mathbf{B}^{(em)} = \mathrm{rot}\mathbf{A}^{(em)}$, and $\mathbf{S} = (\hbar/2)\boldsymbol{\sigma}$ is the spin operator. This Hamiltonian can also be derived from the Dirac equation by means of the substitution $(\varphi,\chi)^{\mathrm{t}} = e^{-i(mc^2/\hbar)t}(\tilde{\varphi},\tilde{\chi})^{\mathrm{t}}$ in (2.3). The second component of (2.3) gives $2imc\tilde{\chi} = \hbar\sigma^k\nabla_k^A\tilde{\varphi} + O(1/c)$, to be inserted into the first. Performing the limit $c\to\infty$, and writing ψ for the 2-spinor $\tilde{\varphi}$, we find,

$$\frac{1}{i}\nabla_0^A\psi - \frac{\hbar}{2m}\sigma^i\nabla_i^A\sigma^k\nabla_k^A\psi = 0. \tag{A.7}$$

This is equivalent to the Pauli equation (A.1), by virtue of the first of the identities mentioned after (A.2). Moreover, if we use the above substitution in the Dirac current (2.4), we recover the non-relativistic current (A.5).

As for the current matrix, it is better to refrain from the Gordon decomposition (A.5), and to directly derive it from the Lagrangian defined in (A.3), or from the Dirac current matrices (2.11) via the indicated substitution. In (A.3), we replace the left conjugated spinor by $\psi_n^\dagger$ and the right one by ψ_m, to find the charge density $\rho_{mn} = q\psi_n^\dagger\psi_m$ and the 3-current matrix,

$$j_{mn}^k = \frac{i\hbar q}{2m}((\sigma^i\nabla_i^A\psi_n)^\dagger\sigma^k\psi_m - \psi_n^\dagger\sigma^k\sigma^i\nabla_i^A\psi_m). \tag{A.8}$$

As we systematically linearize in q, the tachyonic charge, we may replace here ∇_i^A by ∂_i or ∇_i^{em}, cf. after (2.1).

We turn to the helicity operator, the spin operator projected onto the wave vector. In the non-relativistic case, this operator is just $\boldsymbol{\sigma}\mathbf{k}/k$, apart from a factor of one-half, cf. (A.6). The relativistic helicity operator, $\mathbf{k}\boldsymbol{\Sigma}/k$, likewise reduces to $\pm\boldsymbol{\sigma}\mathbf{k}/k$, the signs applying to positive/negative frequency subspaces, respectively, cf. after (2.7). It remains to solve $(\boldsymbol{\sigma}\mathbf{k}-ks)\varphi'_{k,s}=0$, where s takes the values ±1 for the determinant to vanish. A complete set of eigenvectors is easily found,

$$\varphi'_{k,s}=\frac{1}{\sqrt{2k(k+sk_3)}}\begin{pmatrix}k+sk_3\\ s(k_1+ik_2)\end{pmatrix}=\frac{1}{\sqrt{2(1+s\cos\theta_k)}}\begin{pmatrix}1+s\cos\theta_k\\ s\sin\theta_k e^{i\chi_k}\end{pmatrix}. \quad (A.9)$$

(Here, $k=|\mathbf{k}|$, and k_i denotes the respective component of $\mathbf{k}$.) We have chosen the normalization $\varphi'^{\dagger}_{k,s}\varphi'_{k,s}=1$ and a convenient but arbitrary phase. If $\mathbf{k}$ is a coordinate vector $\mathbf{e}_i$, then the normalized solution is $\varphi'_{e_1,s}=2^{-1/2}(1,s)^{\mathrm{t}}$ or $\varphi'_{e_2,s}=2^{-1/2}(1,is)^{\mathrm{t}}$ or $\varphi'_{e_3,s}=(1/2)(1+s,1-s)^{\mathrm{t}}$, respectively, again up to an arbitrary phase factor. In (A.9), we have introduced polar coordinates with polar axis $\mathbf{k}$,

$$k_1+ik_2=k\sin\theta_k e^{i\chi_k},\quad k_3=k\cos\theta_k,$$
$$k\pm sk_3=k(1\pm s\cos\theta_k). \quad (A.10)$$

This is a very efficient parametrization, by virtue of $\sin^2\theta_k=(1+s\cos\theta_k)(1-s\cos\theta_k)$, when calculating products of matrix elements, e.g.,

$$2(\varphi'^{\dagger}_{q,r}\boldsymbol{\sigma}\mathbf{p}\varphi'_{k,s})(\varphi'^{\dagger}_{k,s}\boldsymbol{\sigma}\mathbf{b}\varphi'_{q,r})=i(\mathbf{p}\times\mathbf{b})(r\mathbf{q}_0-s\mathbf{k}_0)+\mathbf{pb}+rs((\mathbf{pk}_0)(\mathbf{bq}_0)+(\mathbf{pq}_0)(\mathbf{bk}_0)-(\mathbf{pb})(\mathbf{k}_0\mathbf{q}_0)). \quad (A.11)$$

The subscript zeros denote unit vectors; the respective (unnormalized) wave vectors $\mathbf{q}$ and $\mathbf{k}$, indicated in the spinor subscripts, define the spinors according to (A.9). (These spinors depend only on the unit vectors, of course.) The spin indices r and s both admit the values ±1. The arbitrary phase factors of the spinors (A.9) cancel each other. This formula can readily be derived for arbitrary real projection vectors $\mathbf{p}$ and $\mathbf{b}$, by means of the above angular parametrization, and it evidently stays valid for complex vectors as well. Other products of matrix elements such as

$$2(\varphi'^{\dagger}_{q,r}\boldsymbol{\sigma}\mathbf{p}\varphi'_{k,s})(\varphi'^{\dagger}_{k,s}\varphi'_{q,r})=\mathbf{p}(s\mathbf{k}_0+r\mathbf{q}_0)+irs\mathbf{p}(\mathbf{k}_0\times\mathbf{q}_0),$$
$$2(\varphi'^{\dagger}_{q,r}\varphi'_{k,s})(\varphi'^{\dagger}_{k,s}\boldsymbol{\sigma}\mathbf{b}\varphi'_{q,r})=\mathbf{b}(s\mathbf{k}_0+r\mathbf{q}_0)-irs\mathbf{b}(\mathbf{k}_0\times\mathbf{q}_0), \quad (A.12)$$

$$2\left|\varphi'^{\dagger}_{k,s}\varphi'_{q,r}\right|^2=1+rs\mathbf{k}_0\mathbf{q}_0,$$
$$2\left|\varphi'^{\dagger}_{k,s}\boldsymbol{\sigma}\varphi'_{q,r}\right|^2=3-rs\mathbf{k}_0\mathbf{q}_0, \quad (A.13)$$

are obtained from (A.11), by identifying (the current projection vectors) $\mathbf{b}$ or $\mathbf{p}$ with the spin projection vectors $\mathbf{q}$ or $\mathbf{k}$, respectively, and by applying the eigenvalue equations defining the spinors, such as $(\boldsymbol{\sigma}\mathbf{q}-qr)\varphi'_{q,r}=0$. The squared matrix elements of the Dirac current can be reduced to those in (A.13), cf. (2.13–2.15), and we also mention $\varphi'^{\dagger}_{k,s}\varphi'_{k,r}=\delta_{sr}$ and $\varphi'^{\dagger}_{k,s}\sigma^i\varphi'_{k,s}=sk_i/k$.

Products of type $(\varphi'^{\dagger}_{q,r}(P+\boldsymbol{\sigma}\mathbf{p})\varphi'_{k,s})(\varphi'^{\dagger}_{k,s}(B+\boldsymbol{\sigma}\mathbf{b})\varphi'_{q,r})$, where B and P are arbitrary complex numbers, can be assembled from (A.11–A.13). The special case $P=B^*$ and $\mathbf{p}=\mathbf{b}^*$ reads,

$$2\left|\varphi'^{\dagger}_{k,s}(B+\boldsymbol{\sigma}\mathbf{b})\varphi'_{q,r}\right|^2=|B|^2(1+rs\mathbf{k}_0\mathbf{q}_0)+\mathbf{bb}^*(1-rs\mathbf{k}_0\mathbf{q}_0)+rs((\mathbf{b}^*\mathbf{k}_0)(\mathbf{bq}_0)+(\mathbf{b}^*\mathbf{q}_0)(\mathbf{bk}_0)+i(B\mathbf{b}^*-B^*\mathbf{b})(\mathbf{k}_0\times\mathbf{q}_0))+(B\mathbf{b}^*+B^*\mathbf{b})(s\mathbf{k}_0+r\mathbf{q}_0)+i(\mathbf{b}\times\mathbf{b}^*)(s\mathbf{k}_0-r\mathbf{q}_0), \quad (A.14)$$

and we note the symmetry,

$$\left|\varphi'^{\dagger}_{k,s}(B+\boldsymbol{\sigma}\mathbf{b})\varphi'_{q,r}\right|^2=\left|\varphi'^{\dagger}_{q,r}(B^*+\boldsymbol{\sigma}\mathbf{b}^*)\varphi'_{k,s}\right|^2. \quad (A.15)$$

Multiple products of Pauli matrices in squared matrix elements can be reduced to (A.14) by applying $(\boldsymbol{\sigma}\mathbf{a})(\boldsymbol{\sigma}\mathbf{b})=\mathbf{ab}+i\boldsymbol{\sigma}(\mathbf{a}\times\mathbf{b})$. In (A.14), we split $B=F_1+iF_2$ and $\mathbf{b}=\mathbf{G}_1+i\mathbf{G}_2$, to find,

$$\left|\varphi'^{\dagger}_{k,s}\left(F_1+iF_2+\boldsymbol{\sigma}\left(\mathbf{G}_1+i\mathbf{G}_2\right)\right)\varphi'_{q,r}\right|^2=\tfrac{1}{2}\left(F_1^2+F_2^2\right)(1+rs\mathbf{k}_0\mathbf{q}_0)+\tfrac{1}{2}\left(|\mathbf{G}_1|^2+|\mathbf{G}_2|^2\right)\times(1-rs\mathbf{k}_0\mathbf{q}_0)+rs\left((\mathbf{G}_1\mathbf{k}_0)(\mathbf{G}_1\mathbf{q}_0)+(\mathbf{G}_2\mathbf{k}_0)(\mathbf{G}_2\mathbf{q}_0)+(F_1\mathbf{G}_2-F_2\mathbf{G}_1)(\mathbf{k}_0\times\mathbf{q}_0)\right)+(F_1\mathbf{G}_1+F_2\mathbf{G}_2)\times(s\mathbf{k}_0+r\mathbf{q}_0)+(\mathbf{G}_1\times\mathbf{G}_2)(s\mathbf{k}_0-r\mathbf{q}_0). \quad (A.16)$$

In the product formulas (A.11–A.16), it is understood that $\mathbf{k}_0$ and $\mathbf{q}_0$ are real unit vectors, defining the 2-spinors in the matrix elements by virtue of (A.9). Matrix elements over 4-spinors can be decomposed into 2-spinor elements of type (A.16), as done in Appendix B, cf. (B.14) and (B.15). In this way, we can save the formalism needed in manifestly covariant spin averaging procedures, such as the introduction of spin and energy projection operators and trace calculations of products of Dirac matrices.

Finally, we return to the non-relativistic spinor current (A.8), and substitute the eigenfunctions, $\psi_n=L^{-3/2}\varphi'_{k_n,s_n}e^{i(\mathbf{k}_n\mathbf{x}-\omega_n t)}$, of the free Pauli equation and the helicity operator. Here, we use the non-relativistic dispersion relation, $\omega_n=\hbar k_n^2/(2m)$, and the spinors φ'_{k_n,s_n} in (A.9). The normalization is such that $\int_{L^3}\rho_{mn}d^3x=q\delta_{mn}$, where ρ_{mn} is the charge density matrix defined before (A.8). By employing the eigenvalue equation for φ'_{k_n,s_n} as stated before (A.9), we find the time separated current matrix,

$$\tilde{\rho}_{mn}=\frac{q}{L^3}\varphi'^{\dagger}_{n}\varphi'_{m}e^{i\mathbf{k}_{mn}\mathbf{x}},$$
$$\tilde{\mathbf{j}}_{mn}=\frac{q\hbar}{2mL^3}(k_m s_m+k_n s_n)\varphi'^{\dagger}_{n}\boldsymbol{\sigma}\varphi'_{m}e^{i\mathbf{k}_{mn}\mathbf{x}}. \quad (A.17)$$

(The time factorization is defined as in (2.13).) We square these matrix elements by means of (A.13), to recover the non-relativistic limit of the spin summation in (2.15),

$$\sum_{s_n=\pm 1} |\tilde{\rho}_{mn}|^2 = \frac{q^2}{L^6},$$
$$\sum_{s_n=\pm 1} \left|\tilde{\mathbf{j}}_{mn}\right|^2 = \frac{q^2}{L^6}\left(\frac{3}{4}\left(\upsilon_m^2+\upsilon_n^2\right) - \frac{1}{2}\boldsymbol{\upsilon}_m\boldsymbol{\upsilon}_n\right), \quad \text{(A.18)}$$

where $\boldsymbol{\upsilon}_m = \hbar\mathbf{k}_m/m$.

Appendix B: Tachyonic ionization cross-sections: matrix elements in the high-energy regime

In (B.1–B.7), we outline the calculation of the Fourier transformed electronic ground state and the scattering state, including the first order corrections in the electric fine structure constant [27]. This is not new, of course, but we need it in tune with the notation and conventions of Section 2 when calculating the current matrix elements, cf. (B.8–B.21). Subsequently, we will explain the angular parametrization of the cross-sections and work out their dependence on the electronic Lorentz factor and the electron-tachyon mass ratio.

First, we settle the bound state. We start with the Dirac Hamiltonian in a Coulomb potential, that is, with H^{em} in (2.7) ($\hbar = c = 1$), and factorize the wave function, $\psi = u_n e^{-i\varepsilon\omega t}$, to arrive at the time separated Dirac equation,

$$\left(\varepsilon\omega + i\alpha^k\partial_k - m\beta + eA_0^{em}\right)u_n = 0. \quad \text{(B.1)}$$

Applying $(\varepsilon\omega - i\alpha^k\partial_k + m\beta + eA_0^{em})$, we find,

$$\left(\Delta + \omega^2 - m^2 + 2\varepsilon\omega eA_0^{em} - ie\alpha^k A_{0,k}^{em} + e^2 A_0^{em2}\right)u_n = 0. \quad \text{(B.2)}$$

We identify $eA_0^{em} = -V$, where $V = -Ze^2/(4\pi r)$, cf. after (2.1), and drop terms of order e^2, that is, the squared potential in (B.2). Finally, we consider positive frequencies, $\varepsilon = 1$, cf. after (2.7), write $\omega = m - E_n$, and introduce $\alpha_Z = Ze^2/(4\pi) \approx Z/137$, to arrive at

$$\left(\frac{1}{2m}\Delta - E_n + \frac{\alpha_Z}{r} - i\frac{\alpha_Z}{2m}\alpha^k\left(\partial_k\frac{1}{r}\right)\right)u_n = 0. \quad \text{(B.3)}$$

In lowest order in α_Z, we can readily guess the solutions, $u_n \approx (1 + i\alpha_Z\boldsymbol{\alpha}\mathbf{r}_0/2)u'_0\psi_n^C$, where ψ_n^C is a bound state wave function (normalized to one) of the Schrödinger equation, obtained by dropping the gradient term in (B.3), the relativistic correction, that is. The $E_n > 0$ are the corresponding Coulomb bound state energies, and $\mathbf{r}_0$ is the unit coordinate vector. The constant rest frame spinor $u'_0 = (\varphi'_{b,r}, 0)^{\mathrm{t}}$, cf. after (2.7), is likewise normalized to unity, $u_0^{\dagger}u'_0 = 1$; the 2-spinor $\varphi'_{b,r}$ is defined in (A.9), where $\mathbf{b}$ is an arbitrary (real, unnormalized) spin projection vector, and $r = \pm 1$ is the spin index. It is easy to check that these u_n are solutions of (B.3), since $\psi_{n,k}^C/\psi_n^C$ is of $\mathrm{O}(\alpha_Z)$ for Coulomb bound states. We will focus on the ground state, $E_0 = m\alpha_Z^2/2$, with the normalized eigenfunction $\psi_0^C = (\pi a_B^3)^{-1/2}e^{-r/a_B}$, where $a_B = 1/(m\alpha_Z)$. To facilitate the subsequent partial integrations, we write the ground state of (B.3) as $u_{0,r} \approx (1 - (i/(2m))\alpha^k\partial_k)u'_{0,r}\psi_0^C$, where we have explicitly indicated the spin index r of the 2-spinor $\varphi'_{b,r}$ in u'_0. We will need this wave function in momentum space; the Fourier transform of $u_{0,r}$ reads,

$$\hat{u}_{0,r}(\mathbf{q}) \approx \left(1 + \frac{\boldsymbol{\alpha}\mathbf{q}}{2m}\right)u'_{0,r}\hat{\psi}_0^C(\mathbf{q}),$$
$$\hat{\psi}_0^C(\mathbf{q}) = \frac{8\pi}{a_B}\frac{\left(\pi a_B^3\right)^{-1/2}}{\left(\mathbf{q}^2 + a_B^{-2}\right)^2}. \quad \text{(B.4)}$$

(The convention for Fourier transforms is $\varphi(\mathbf{x}) = (2\pi)^{-3}\int\hat{\varphi}(\mathbf{q})e^{i\varepsilon\mathbf{q}\mathbf{x}}d^3q$, and we will restrict to positive frequencies, $\varepsilon = 1$.) In deriving (B.4), we used $\int e^{-\mu r - i\mathbf{q}\mathbf{x}}d^3x = 8\pi\mu(\mathbf{q}^2+\mu^2)^{-2}$, which is a limit definition, $(2\pi)^3\delta(\mathbf{q}, \mu\to 0)$, of the Dirac function. This suggests to approximate, $\hat{\psi}_0^C(\mathbf{q}) \approx (2\pi)^3\delta(\mathbf{q})(\pi a_B^3)^{-1/2}$, if permissible, cf. (B.10).

We turn to the electronic scattering state. After time separation, $\psi = u_f e^{-i\varepsilon\omega t}$, we use the ansatz $u_f = u'_f e^{i\varepsilon\mathbf{p}\mathbf{x}} + u_f^C(\mathbf{x})$ in (B.1), where $u'_f(\mathbf{p}) = (\varphi_{p,s}, \chi_{p,s})^{\mathrm{t}}$ is a constant spinor defined in (2.9) or (2.10). The spinor u_f is supposed to solve the field equation (B.1) with $eA_0^{em} = \alpha_Z e^{-\mu r}/r$; here, an exponential has been inserted to regularize the Fourier transform of u_f, the limit $\mu\to 0$ will be carried out when appropriate, cf. (B.7). The constant spinor $u'_f(\mathbf{p})$ satisfies $(\varepsilon\omega - \varepsilon\boldsymbol{\alpha}\mathbf{p} - m\beta)u'_f = 0$, with $\mathbf{p}^2 = \omega^2 - m^2$, cf. after (2.12). The Coulomb correction u_f^C in the above ansatz is thus obtained by solving, cf. (B.1) and (B.2),

$$\left(\varepsilon\omega + i\alpha^k\partial_k - m\beta\right)u_f^C = -\alpha_Z r^{-1}e^{-\mu r + i\varepsilon\mathbf{p}\mathbf{x}}u'_f,$$
$$\left(\Delta + \omega^2 - m^2\right)u_f^C = -\alpha_Z(\varepsilon\omega - i\alpha^k\partial_k + m\beta) \times (r^{-1}e^{-\mu r + i\varepsilon\mathbf{p}\mathbf{x}})u'_f. \quad \text{(B.5)}$$

We again restrict to $\varepsilon = 1$. The second order equation is most easily dealt with in Fourier space. We multiply with $e^{-i\mathbf{q}\mathbf{x}}$, and integrate over 3-space. By making use of $\int r^{-1}e^{-\mu r - i\mathbf{p}\mathbf{x}}d^3x = 4\pi(\mathbf{p}^2+\mu^2)^{-1}$, we find,

$$(\omega^2 - m^2 - \mathbf{q}^2)\hat{u}_f^C(\mathbf{q}) = -4\pi\alpha_Z\frac{\omega + m\beta + \boldsymbol{\alpha}\mathbf{q}}{(\mathbf{p}-\mathbf{q})^2 + \mu^2}u'_f(\mathbf{p}), \quad \text{(B.6)}$$

where $\hat{u}_f^C(\mathbf{q}) = \int u_f^C(\mathbf{x})e^{-i\mathbf{q}\mathbf{x}}d^3x$. We can thus assemble the Fourier transform of the final electronic state $u_{f,s}$ (with the spin index s now explicitly indicated),

$$\hat{u}_{f,s}(\mathbf{q}) = (2\pi)^3\delta(\mathbf{p}-\mathbf{q})u'_{f,s}(\mathbf{p}) + \hat{u}_{f,s}^C(\mathbf{q}),$$
$$\hat{u}_{f,s}^C(\mathbf{q}) \approx -4\pi\alpha_Z\frac{2\omega - \boldsymbol{\alpha}(\mathbf{p}-\mathbf{q})}{(\mathbf{p}^2-\mathbf{q}^2)(\mathbf{p}-\mathbf{q})^2}u'_{f,s}(\mathbf{p}). \quad \text{(B.7)}$$

The spin is projected onto the (asymptotic) wave vector $\mathbf{p}$, so that $u'_{f,s}(\mathbf{p}) = (\varphi_{p,s}, \chi_{p,s})^{\mathrm{t}}$, as defined by (2.9) and (A.9). We have finally put $\mu = 0$, as no regularization is required in the following integrations.

Some comments on box quantization and the normalization convention for scattering states are in order. Continuum solutions of the Dirac equation (free or in a Coulomb potential) are normalized according to $\int_{R^3} u^\dagger_{p,r} u_{k,s} d^3x = (2\pi)^3 \delta(\mathbf{k}-\mathbf{p})\delta_{rs}$, which means that asymptotically $u_{p,s} \sim u'_{p,s} e^{i\varepsilon \mathbf{p}\mathbf{x}}$, where $u'_{p,s}$ is a constant spinor (2.9) or (2.10). In box quantization, we have to switch from lattice summations over $\mathbf{p} = 2\pi\mathbf{n}/L$ to the continuum limit, $L^3(2\pi)^{-3}d^3\mathbf{p}$, and vice versa. The free modes of the continuous spectrum (normalized as indicated) are discretized by replacing $\mathbf{p} \to 2\pi\mathbf{n}/L$, and adding a factor of $L^{-3/2}$ according to (2.12). The same applies to the perturbed (Coulomb) wave fields. As for bound states, we always normalize them by $\int_{R^3} u^\dagger_{i,r} u_{j,s} d^3x = \delta_{ij}\delta_{rs}$, even when invoking box quantization, as they are already discrete. That is, we perform the continuum limit when normalizing, even though we keep the box size finite otherwise.

We turn to the actual calculation of the matrix elements. In the ionization cross-sections (3.19), the squared elements $\left|J^{T,L}_{mr,ns}\right|^2$ enter, where

$$J^T_{mr,ns} := \int \boldsymbol{\varepsilon}_{\mathbf{k},\lambda} \mathbf{j}_{mr,ns} e^{i\mathbf{k}\mathbf{x}} d^3x,$$

$$J^L_{mr,ns} := \int \rho_{mr,ns} e^{i\mathbf{k}\mathbf{x}} d^3x. \tag{B.8}$$

The $\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}$ are real (linear) polarization unit vectors of the tachyon field as defined after (3.3). The current matrices $\rho_{mr,ns} = q u^\dagger_{ns} u_{mr}$ and $\mathbf{j}_{mr,ns} = q u^\dagger_{ns} \boldsymbol{\alpha} u_{mr}$ were introduced after (3.2) ($\hbar = c = 1$), and the matrix vector $\boldsymbol{\alpha}$ is defined in (2.6), in standard representation. We identify u_{mr} with the initial electronic bound state $u_{0,r}$ in (B.4), and u_{ns} with the scattering state $u_{f,s}$ in (B.7). This identification is arbitrary and can be interchanged in the squared matrix elements. We rewrite the transversal current (B.8) in Fourier transforms,

$$J^T_{r,s} = \frac{q}{(2\pi)^3} \int \hat{u}^\dagger_{f,s}(\mathbf{q}) \boldsymbol{\alpha}\boldsymbol{\varepsilon}_{\mathbf{k},\lambda} \hat{u}_{0,r}(\mathbf{q}-\mathbf{k}) d^3q, \tag{B.9}$$

and substitute the initial and final states (B.4) and (B.7). The indices r and s in $J^T_{r,s}$ stand for the initial and final spin states, respectively. The d^3q-integration gets trivial by virtue of δ-functions,

$$J^T_{r,s} \approx q u'^\dagger_{f,s}(\mathbf{p}) \boldsymbol{\alpha}\boldsymbol{\varepsilon}_{\mathbf{k},\lambda} \left(1 + \frac{\boldsymbol{\alpha}(\mathbf{p}-\mathbf{k})}{2m}\right) u'_{0,r} \hat{\psi}^C_0(\mathbf{p}-\mathbf{k}) + \frac{q}{\sqrt{\pi a_B^3}} \hat{u}^{C\dagger}_{f,s}(\mathbf{k}) \boldsymbol{\alpha}\boldsymbol{\varepsilon}_{\mathbf{k},\lambda} u'_{0,r}. \tag{B.10}$$

The longitudinal current $J^L_{r,s}$ is likewise given by (B.10), but with the matrix $\boldsymbol{\alpha}\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}$ dropped in both terms. Inserting here the explicit formulas for $\hat{\psi}^C_0$ and $\hat{u}^C_{f,s}$ as stated in (B.4) and (B.7), we find,

$$J^{T,L}_{r,s} \approx \frac{8\pi q}{(\pi a_B^3)^{1/2} a_B} \frac{u'^\dagger_{f,s}(\mathbf{p}) j^{T,L} u'_{0,r}}{(\mathbf{p}-\mathbf{k})^2}, \tag{B.11}$$

where $a_B := 1/(m\alpha_Z)$, and the $j^{T,L}$ are assembled as,

$$j^T := A\boldsymbol{\alpha}\boldsymbol{\varepsilon}_{\mathbf{k},\lambda} + (\boldsymbol{\alpha}\boldsymbol{\varepsilon}_{\mathbf{k},\lambda})(\boldsymbol{\alpha}\mathbf{B}) + (\boldsymbol{\alpha}\mathbf{C})(\boldsymbol{\alpha}\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}),$$
$$j^L := A + \boldsymbol{\alpha}(\mathbf{B}+\mathbf{C}), \tag{B.12}$$

$$A := \frac{1}{(\mathbf{p}-\mathbf{k})^2} - \frac{\omega}{m}\frac{1}{p^2-k^2}, \quad \mathbf{B} := \frac{\mathbf{p}-\mathbf{k}}{2m(\mathbf{p}-\mathbf{k})^2},$$
$$\mathbf{C} := \frac{\mathbf{p}-\mathbf{k}}{2m(p^2-k^2)}. \tag{B.13}$$

Here, $\mathbf{p}$ and $\mathbf{k}$ are the asymptotic wave vectors of the ejected electron and the incident ionizing tachyon, respectively. The dispersion relations read $p^2 = \omega_p^2 - m^2$ and $k^2 = \omega_k^2 + m_t^2$, cf. after (3.14). From now on, we write the electronic and tachyonic frequencies with p and k subscripts; ω in (B.13) and in all other formulas above is thus the electronic ω_p. The rest frame spinor $u'_{0,r} = (\varphi'_{b,r}, 0)^{\mathrm{t}}$ in (B.11) is defined with a 2-spinor (A.9), where the spin projection vector $\mathbf{b}$ is arbitrary, cf. after (B.3). The asymptotic amplitude $u'_{f,s}(\mathbf{p})$ of the outgoing wave, cf. before (B.5), is composed of two 2-spinors, $(\varphi_{p,s}, \chi_{p,s})^{\mathrm{t}}$, defined by (2.9) and (A.9), with spin projection onto the electronic wave vector $\mathbf{p}$.

The transversal matrix element in (B.11) can be reduced to a 2-spinor element,

$$u'^\dagger_{f,s}(\mathbf{p}) j^T u'_{0,r} = \frac{\varphi'^\dagger_{p,s}(F^T + \boldsymbol{\sigma}(\mathbf{G}^T_1 + i\mathbf{G}^T_2))\varphi'_{b,r}}{\sqrt{2\omega_p(\omega_p+m)}}, \tag{B.14}$$

$$F^T := (\omega_p+m)(\mathbf{B}+\mathbf{C})\boldsymbol{\varepsilon}_{\mathbf{k},\lambda},$$
$$\mathbf{G}^T_1 := spA\boldsymbol{\varepsilon}_{\mathbf{k},\lambda},$$
$$\mathbf{G}^T_2 := (\omega_p+m)\boldsymbol{\varepsilon}_{\mathbf{k},\lambda} \times (\mathbf{B}-\mathbf{C}). \tag{B.15}$$

Here, we just multiplied out the transversal 4-spinor element (B.11), using the indicated 2-spinor decomposition of the initial and final state, the standard representation (2.6) of $\boldsymbol{\alpha}$ in (B.12), and the reduction formula for products of Pauli matrices stated after (A.15). Similarly, the longitudinal 4-spinor element in (B.11) reduces to

$$u'^\dagger_{f,s}(\mathbf{p}) j^L u'_{0,r} = \frac{\varphi'^\dagger_{p,s}(F^L + \boldsymbol{\sigma}\mathbf{G}^L)\varphi'_{b,r}}{\sqrt{2\omega_p(\omega_p+m)}}, \tag{B.16}$$

$$F^L := (\omega_p+m)A, \quad \mathbf{G}^L := sp(\mathbf{B}+\mathbf{C}). \tag{B.17}$$

The primed 2-spinors in the matrix elements (B.14) and (B.16) are defined in (A.9), and the (real) coefficients A, $\mathbf{B}$ and $\mathbf{C}$ are stated in (B.13). These elements can be squared by means of (A.16), if we there identify $\mathbf{k}_0$ and $\mathbf{q}_0$ with the unit vectors $\mathbf{p}_0$ and $\mathbf{b}_0$, respectively. The squared matrix elements are clumsy, but they simplify when averaged over the initial spin index r and summed over the final spin s, according to

$$\langle j^{T,L}\rangle^2 := \frac{1}{2}\sum_{r,s=\pm 1} \left|u'^\dagger_{f,s}(\mathbf{p}) j^{T,L} u'_{0,r}\right|^2. \tag{B.18}$$

As noted, the matrix elements (B.14) and (B.16) are special cases of the element squared in (A.16), and we readily find, after performing the summation (B.18),

$$\omega_p\left(\omega_p+m\right)\langle j^T\rangle^2=\tfrac{1}{2}\left(F^{T2}+\left|\mathbf{G}_1^T\right|^2+\left|\mathbf{G}_2^T\right|^2\right)+s\mathbf{p}_0\left(\mathbf{G}_1^T F^T+\mathbf{G}_1^T\times\mathbf{G}_2^T\right),$$
$$\omega_p\left(\omega_p+m\right)\langle j^L\rangle^2=\tfrac{1}{2}\left(F^{L2}+\left|\mathbf{G}^L\right|^2\right)+s\mathbf{p}_0\mathbf{G}^L F^L. \quad \text{(B.19)}$$

The right-hand side is independent of the spin variable s, which drops out when substituting (B.15) and (B.17), since $s^2=1$. In this way, we arrive at,

$$\langle j^T\rangle^2=\frac{1}{2\omega_p m}\left[(A\mathbf{p}-(\omega_p+m)(\mathbf{B}-\mathbf{C}))^2\frac{m}{\omega_p+m}+4m\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\mathbf{B}(A\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\mathbf{p}+(\omega_p+m)\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\mathbf{C})\right], \quad \text{(B.20)}$$

$$\langle j^L\rangle^2=\frac{1}{2\omega_p(\omega_p-m)}(A\mathbf{p}+(\omega_p-m)(\mathbf{B}+\mathbf{C}))^2, \quad \text{(B.21)}$$

where the shortcuts A, $\mathbf{B}$ and $\mathbf{C}$ are defined in (B.13) (with ω in A replaced by ω_p), and the dispersion relations stated after (B.13) apply.

We still need to find a parametrization of the matrix elements (B.20) and (B.21) that renders them more accessible. To this end, we substitute $\omega_p=m\gamma$, cf. after (3.18), and define the shortcuts $\mathbf{X}^{T,L}$ by

$$A\mathbf{p}-(\omega_p+m)(\mathbf{B}-\mathbf{C})=:(\mathbf{p}-\mathbf{k})^{-2}(p^2-k^2)^{-1}\mathbf{X}^T,$$
$$A\mathbf{p}+(\omega_p-m)(\mathbf{B}+\mathbf{C})=:(\mathbf{p}-\mathbf{k})^{-2}(p^2-k^2)^{-1}\mathbf{X}^L, \quad \text{(B.22)}$$

so that, cf. (B.13),

$$\mathbf{X}^T=(\gamma-1)(\mathbf{pk}-p^2)\mathbf{p}+(\gamma+1)(\mathbf{pk}-k^2)\mathbf{k},$$
$$\mathbf{X}^L=(\gamma+1)(\mathbf{pk}-k^2)\mathbf{p}+(\gamma-1)(\mathbf{pk}-p^2)\mathbf{k}. \quad \text{(B.23)}$$

We introduce polar coordinates with $\mathbf{k}$ as polar axis, $\mathbf{pk}=pk\cos\theta$, and parametrize the absolute values as

$$p^2=m^2(\gamma^2-1), \quad k^2=m^2(\gamma-1)^2\eta^2, \quad \text{(B.24)}$$

where $\eta\geq 1$ is a dimensionless parameter to be specified by energy conservation. In this way, the squares of the vectors (B.23) admit the parametrization,

$$\left|\mathbf{X}^T\right|^2=m^6(\gamma-1)^5(\gamma+1)^2\Big\{(\eta^2-1)^2a(\theta)+\eta^2\sin^2\theta\left[2(1-\eta^2)+a(\theta)\right]\Big\},$$
$$\left|\mathbf{X}^L\right|^2=m^6\eta^2(\gamma-1)^4(\gamma+1)\Big\{4a(\theta)+\sin^2\theta[2\eta^2(\gamma-1)^2-2(\gamma+1)^2+(\gamma^2-1)a(\theta)]\Big\}, \quad \text{(B.25)}$$

where we have introduced the shortcut,

$$a(\theta):=\gamma+1+(\gamma-1)\eta^2-2\eta\sqrt{\gamma^2-1}\cos\theta. \quad \text{(B.26)}$$

The second term in (B.20) depends on the transversal polarization vectors, which satisfy $\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\mathbf{k}=0$, $\lambda=1,2$. This allows the polar parametrization $\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\mathbf{p}=p\sin\theta\cos\varphi$. Here, we specify a linear transversal polarization, say, $\lambda=1$, for the incident tachyons, but this is not a restriction, as $\boldsymbol{\varepsilon}_{\mathbf{k},1}$ can be arbitrarily chosen, cf. after (3.3). Analogously to (B.22), we introduce the shortcut Y^T, cf. (B.20),

$$4m\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\mathbf{B}(A\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\mathbf{p}+(\omega_p+m)\boldsymbol{\varepsilon}_{\mathbf{k},\lambda}\mathbf{C})=:(\mathbf{p}-\mathbf{k})^{-4}(p^2-k^2)^{-1}Y^T, \quad \text{(B.27)}$$

which is easily seen to admit the parametrization,

$$Y^T=m^4(\gamma-1)^2(\gamma+1)\sin^2\theta\cos^2\varphi\times\left[2(\gamma+1)-2(\gamma-1)\eta^2-(\gamma-1)a(\theta)\right], \quad \text{(B.28)}$$

where $a(\theta)$ is defined in (B.26). The two remaining factors on the right-hand side in (B.22) and (B.27), are parametrized as, cf. (B.24),

$$p^2-k^2=2m^2(\gamma-1)\delta,$$
$$2\delta:=\gamma+1-(\gamma-1)\eta^2,$$
$$(\mathbf{p}-\mathbf{k})^2=m^2(\gamma-1)a(\theta). \quad \text{(B.29)}$$

When restoring units, a factor of $c^2/\hbar^2$ has to be added to the mass-squares in (B.29). To proceed further, we need to specify the parameters η and δ. Energy conservation, as discussed after (3.18), is invoked to relate these parameters to the electronic Lorentz factor and the tachyon-electron mass ratio.

By virtue of (B.22–B.29) and (3.24), we can reassemble the matrix elements (B.20) and (B.21) as,

$$\langle j^{T,L}\rangle^2=\frac{\hbar^4(\gamma+1)}{2c^6m^4\delta^2\gamma(\gamma-1)a^2(\theta)}\left(\Sigma^T,\frac{\eta^2}{4}\Sigma^L\right), \quad \text{(B.30)}$$

where we have introduced the dimensionless shortcuts $\Sigma^T(\theta,\varphi)$ and $\Sigma^L(\theta)$ stated in (3.21) and (3.23). The dimensionless parameters η and δ are defined in (3.24) and the angular factor $a(\theta)$, likewise dimensionless, in (B.26) and (3.25). We have restored the units, so that $\langle j^{T,L}\rangle^2\sim L^2t^2$, where $\omega_p\sim 1/t$ and $p,k\sim 1/L$. The spin averages $\langle J^{T,L}\rangle^2$ of the matrix elements $\left|J^{T,L}_{r,s}\right|^2$ in (B.11) are defined after (3.19); they connect to the averages $\langle j^{T,L}\rangle^2$ in (B.30) as,

$$\langle J^{T,L}\rangle^2=\frac{1}{2}\sum_{r,s=\pm1}\left|J^{T,L}_{r,s}\right|^2=\frac{64\pi q^2\alpha_Z^5c^7m^5}{L^3\hbar^5(\mathbf{p}-\mathbf{k})^4}\left(c^2\langle j^T\rangle^2,\langle j^L\rangle^2\right), \quad \text{(B.31)}$$

which readily follows from (B.18). Here, we substitute $(\mathbf{p}-\mathbf{k})^2$ parametrized as in (B.29). The units have been restored so that $\langle J^{T,L}\rangle^2\sim(c^2q^2,q^2)$, according to (B.8). The wave vectors $\mathbf{p}$ and $\mathbf{k}$ have the dimension of inverse length (the units in (B.29) have to be restored accordingly), and the charge q^2 relates to the dimensionless tachyonic fine structure constant as $\alpha_q = q^2/(4\pi\hbar c)$. The dimension of $\langle j^{T,L}\rangle^2$ is $\mathrm{cm}^2\,\mathrm{s}^2$, already restored in (B.30), and the factor L^{-3} in (B.31) stems from box quantization, as pointed out in the remarks preceding (B.8). The differential cross-sections (3.19) thus read,

$$d\sigma^T = \frac{32\alpha_q\alpha_Z^5 c^4 m^2}{\hbar^2\eta}\frac{\gamma\sqrt{\gamma^2-1}\,\langle j^T\rangle^2}{(\gamma-1)^3 a^2(\theta)}d\mathbf{\Omega}, \tag{B.32}$$

$$d\sigma^L = \frac{32\alpha_q\alpha_Z^5 c^4 m_t^2}{\hbar^2\eta^3}\frac{\gamma\sqrt{\gamma^2-1}\,\langle j^L\rangle^2}{(\gamma-1)^5 a^2(\theta)}d\mathbf{\Omega}, \tag{B.33}$$

where we use η in (3.24) , $a(\theta)$ in (B.26), and the matrix elements $\langle j^{T,L}\rangle^2$ in (B.30). This is summarized in (3.20–3.25).

The total cross-sections are stated in (3.26) and (3.27). The solid angle integration, over $d\mathbf{\Omega} = \sin\theta d\theta d\varphi$ in (B.32) and (B.33), is elementary, but the ensuing algebra is lengthy. The azimuthal integration of the transversal section amounts to

$$\frac{1}{2\pi}\int_0^{2\pi}\Sigma^T d\varphi = \frac{a(\theta)}{4(\gamma-1)^2}\frac{m_t^4}{m^4} + \sin^2\theta\left[1 - \frac{1}{2}\frac{\gamma+1}{\gamma-1}\frac{m_t^2}{m^2} - \frac{1}{4(\gamma-1)^2}\frac{m_t^4}{m^4} + \frac{a(\theta)}{4}\left((\gamma-1)(\gamma-2) + \frac{3}{2}\frac{m_t^2}{m^2}\right)\right], \tag{B.34}$$

and the polar integration in the transversal as well as longitudinal section boils down to three integrals,

$$\int_0^{\pi}\frac{\sin\theta d\theta}{a^3(\theta)} = \frac{\gamma\chi}{4\delta^4}, \quad \int_0^{\pi}\frac{\sin^3\theta d\theta}{a^4(\theta)} = \frac{1}{12\delta^4},$$

$$\int_0^{\pi}\frac{\sin^3\theta d\theta}{a^3(\theta)} = \frac{1}{4\eta^2}\frac{1}{\gamma^2-1}\left(\frac{\gamma\chi}{\delta^2} - \frac{c}{2\eta\upsilon\gamma}\log\frac{1+\eta\upsilon/(\chi c)}{1-\eta\upsilon/(\chi c)}\right), \tag{B.35}$$

where the shortcuts χ, δ and η are defined in (3.24) and (3.25).

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